MexiCOPAS

Mexican Cosmology Particles and Strings Schools

Introduction to the Standard Model

MEXICOPAS 2019

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Outline

1. Fundamentals
   - Symmetries in Classical and Quantum Mechanics.
   - Irreducible representations (irreps) of $SU(2)$.
   - Irreps of the HLG: Chirality, Parity and Dirac Equation.
   - Quantum Field theory: complex scalar field.
2. Electroweak interactions: Glashow-Weinberg-Salam theory.
   - Minimal coupling principle in classical mechanics.
   - Gauge theories: Abelian and non-Abelian.
   - Quantum Electrodynamics
   - Fermi theory, IVB theory, parity violation and V-A structure of weak interactions
   - GWS Theory. Spontaneous Breaking of Symmetries.
3. Strong interactions:QCD.
   - Irreducible representations of $SU(3)$
   - Classification of hadrons: Eightfold Way, Quark Model
   - Gauge theory of strong interactions: QCD.
   - Running of couplings: Confinement and asymptotic freedom.
   - Experimental evidence for color degrees of freedom.

What these lectures are...

In these lectures I will give an introduction to :

1. What is an elementary particle? Quantum realm. Symmetries. Group theory.
2. How do we describe their electromagnetic, weak and strong interactions?
3. No time for a historical account on how we arrived at the SM. Fascinating story of hundreds of experiments, a plethora of ideas and more than 40 Nobel prizes during the last century.
4. The most impressive intelectual construction of human kind so far in my opinion.
5. Overconstrained framework after the discovery of the Higgs, but not the end of the story. Many things to do in HEP.
6. Most of the previously explored routes are not appealing now, but new challenges in HEP: neutrinos, dark matter, matter-antimatter asymmetry (CP violation), dark energy. Quantum theory of gravity.

The magic word in physics is...

S Y M M E T R Y

You must master the natural language of symmetries: group theory.

Nature is beautiful... because of symmetry.

Nature is beautiful... because of symmetry ($d \approx 10^{-20} m$).

Modelo Estándar

(0,0)

Boson spin=0	Mass (GeV)
H Higgs	125.6

$SU(3)_C \times SU(2)_L \times U(1)_Y$

Leptons spin = 1/2		
Flavor	Mass GeV/c²	Electric charge
ν_e electron neutrino	$<1\times10^{-8}$	0
e electron	0.000511	−1
ν_μ muon neutrino	<0.0002	0
μ muon	0.106	−1
ν_τ tau neutrino	<0.02	0
τ tau	1.7771	−1

Quarks spin = 1/2		
Flavor	Approx. Mass GeV/c²	Electric charge
u up	0.003	2/3
d down	0.006	−1/3
c charm	1.3	2/3
s strange	0.1	−1/3
t top	175	2/3
b bottom	4.3	−1/3

(1/2,0), (0,1/2)

Strong (color) spin = 1		
Name	Mass GeV/c²	Electric charge
g gluon	0	0

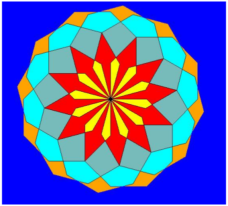
Unified Electroweak spin = 1		
Name	Mass GeV/c²	Electric charge
γ photon	0	0
W^-	80.4	−1
W^+	80.4	+1
Z^0	91.187	0

(1/2,1/2)

Standard Model of Particle Interactions

Quantum theory of electromagnetic, weak and strong interactions of the known elementary particles.

Symmetry $\equiv$ **Invariance** of a **physical system** (**S**) under some **transformations** (**T**)



$$S \xrightarrow{T} S_T = S$$

$$T = \{R_n \equiv R(n\theta)\}, \quad n = 0, 1, ..., 11; \quad \theta = 30^\circ$$

1. $R_n R_m = R_{n+m}$: **Closure**.
2. $R_n(R_m R_l) = (R_n R_m) R_l$: **Associativity**.
3. $R_m R_0 = R_0 R_m = R_m$: **Identity**.
4. $R_n R_{12-n} = R_0$: **Inverse**.

These axioms define a Group.

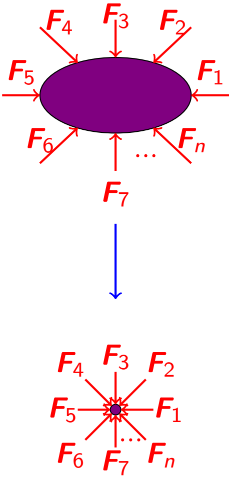
The mathematical language to describe symmetries is **Group Theory** .

Warming up: Classical description of systems

- Primary concepts of inertial frames, forces and interactions

$$\sum_i \boldsymbol{F}_i^{ext} = \frac{d\boldsymbol{P}}{dt}.$$

- First approximation: long distance (low energy) description. Particle physics.
- Conservative forces : $\boldsymbol{F}_i = -\boldsymbol{\nabla} V(\boldsymbol{r}, t)$.
- For some systems conserved quantities emerge (energy, momentum, etc).
- Least action principles $\delta S \equiv \delta \int dt L(q_i, \dot{q}_i, t) = 0$: Lagrange, Hamilton, ...



Symmetries in classical mechanics.

- A guide to the most convenient choice of generalized coordinates, e.g. solution to the central potential $V = V(r)$ is easier taking $q_1 = r$, $q_2 = \theta$, $q_3 = \phi$.
- Independence of V on some generalized coordinate is a symmetry of the system.
- These symmetries yield a conserved quantities

$$p_i = \frac{\partial L}{\partial \dot{q}_i}.$$

Systematics: Emily Noether, 1915

Symmetries of $S = \int dt L(q_i, \dot{q}_i, t) \Leftrightarrow$ conserved quantities

Symmetries and conserved quantities: Noether's theorem

Simplest version: symmetries of L

- In the Lagrange formalism

$$L = \sum_i \frac{1}{2} m\dot{q}_i^2 - V(q_i, t)$$

- Perform a transformation Δq_i

$$\begin{aligned}\Delta L &= \frac{\partial L}{\partial q_i}\Delta q_i + \frac{\partial L}{\partial \dot{q}_i}\Delta \dot{q}_i \\ &= \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i})\Delta q_i + \frac{\partial L}{\partial \dot{q}_i}\Delta \dot{q}_i \\ &= \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}\Delta q_i).\end{aligned}$$

- If L is invariant ($\Delta L = 0$) $\Rightarrow \frac{\partial L}{\partial \dot{q}_i}\Delta q_i = cte.$
- Continuous transformations can be written in terms of some independent parameters ϵ_k

$$\Delta q_i = \sum_k \epsilon_k \xi_{ik}(q, \dot{q})$$

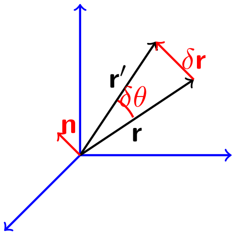
- Conserved Noether charges

$$Q_k = \frac{\partial L}{\partial \dot{q}_i}\xi_{ik}(q, \dot{q}).$$

- Space translations :

$$\delta q_i = \epsilon_k \delta_{ik} \Rightarrow Q_k = \frac{\partial L}{\partial \dot{q}_i}\delta_{ik} = \frac{\partial L}{\partial \dot{q}_k} \equiv p_k$$

- Rotations



$$\delta \boldsymbol{r} = \boldsymbol{n} \times \boldsymbol{r}\delta\theta$$

$$\delta x_i = \varepsilon_{ikl} n_k x_l \delta\theta,$$

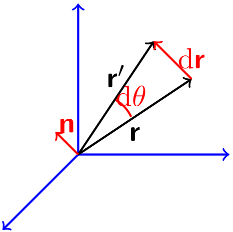
$$\xi_{ik} = \varepsilon_{ikl} x_l, \quad \epsilon_k = n_k \delta\theta$$

$$\Rightarrow Q_k = \frac{\partial L}{\partial \dot{x}_i}\varepsilon_{ikl} x_l = (\boldsymbol{r} \times \boldsymbol{p})_k = \boldsymbol{L}_k$$

Solving the classical dynamics of a particle amounts to find all the symmetries of the system

But elementary particles live in the Quantum Realm

In preparation: Rotations and the exponential map.



$$d\boldsymbol{r} = \boldsymbol{n} \times \boldsymbol{r}\, d\theta$$

$$\begin{aligned} x_i' &= x_i + \epsilon_{ijk} n_j x_k d\theta \\ &= (\delta_{ik} + \epsilon_{ijk} n_j d\theta) x_k \\ &= (\delta_{ik} - i(\boldsymbol{J}\cdot\boldsymbol{n})_{ik} d\theta) x_k \end{aligned}$$

$$(J_i)_{jk} = -i\epsilon_{ijk}$$

Classical mechanics

- Finite rotations ($d\theta \leftrightarrow \frac{\theta}{N}$)

$$\begin{aligned} \boldsymbol{r}' &= R(\boldsymbol{n},\theta)\boldsymbol{r}, \\ R(\boldsymbol{n},\theta) &= \lim_{N\to\infty} (1_{3\times 3} - i(\boldsymbol{J}\cdot\boldsymbol{n})\frac{\theta}{N})^N \\ &= e^{-i\boldsymbol{J}\cdot\boldsymbol{n}\,\theta}. \end{aligned}$$

- R depend continuously on three parameters $\boldsymbol{\theta} = \boldsymbol{n}\theta$.
- $\boldsymbol{J} \equiv$ Rotation generators in $\boldsymbol{R}^3$
- 3×3 matrices satisfying

$$[J_i, J_j] = i\epsilon_{ijk} J_k$$

- $\boldsymbol{J}$ are Hermitian matrices of null trace

$$((J_i)^\dagger)_{jk} = (-i\epsilon_{ikj})^* = -i\epsilon_{ijk} = (J_i)_{jk}$$
$$Tr(J_i) = -i\epsilon_{ijj} = 0$$

- Rotation operators are orthogonal matrices

$$[R(\boldsymbol{n},\theta)]^t = [e^{-i\boldsymbol{J}\cdot\boldsymbol{n}\,\theta}]^t = e^{-i\boldsymbol{J}^t\cdot\boldsymbol{n}\theta} = e^{+i\boldsymbol{J}\cdot\boldsymbol{n}\theta} = [R(\boldsymbol{n},\theta)]^{-1},$$

- with unit determinant

$$Det[R(\boldsymbol{n},\theta)] = Det[e^{-i\boldsymbol{J}\cdot\boldsymbol{n}\,\theta}] = e^{-iTr(\boldsymbol{J})\cdot\boldsymbol{n}} = 1.$$

Rotation operators in classical mechanics (R^3) form a group with the conventional matrix product

Group of 3×3 orthogonal matrices of unit determinant $\equiv SO(3)$ group. Fully characterized by the Lie algebra

$$[J_i, J_j] = i\epsilon_{ijk}J_k$$

Elementary particles live in the Quantum Realm

Quantum Mechanics in a nutshell

1. Physical information of the system encoded in the state vector $|\psi\rangle \in \mathcal{H}$.
2. Observables represented by hermitian operators acting on $\mathcal{H}$ [1].
3. A measurement of the observable A can only yield one of the eigenvalues a_k of the representing operador ($A|a_i\rangle = a_i|a_i\rangle$).
4. Before measuring A we don't know with certainty the resulting eigenvalue a_k, we only know the probability

$$P(a_k) = |\langle a_k|\psi\rangle|^2.$$

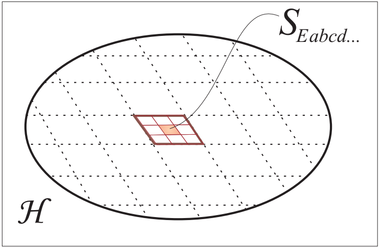
5. If the resulting value of a measurement of A is a_m, the state describing the system immediately after is $|a_m\rangle$.
6. Dynamics is dictated by the Schrodinger equation

$$i\hbar\partial_t|\psi(t)\rangle = H|\psi(t)\rangle.$$

[1] The eigenvalues of a Hermitian operators are real numbers.

Characterizing the Hilbert space

- We need a complete set of commuting observables (CSCO).
- We always choose H in this set $\equiv \{H, A, B, C...\}$.
- Common eigenstates $\{|E, a, b, c, ...\rangle\}$ form a basis of $\mathcal{H}$.

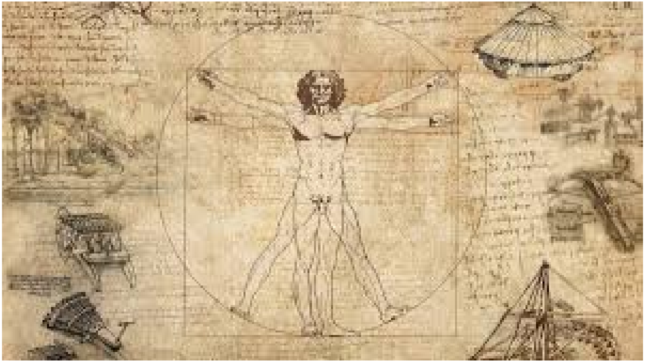


- The eigenvalues of the CSCO, $\{E, a, b, c...\}$ are referred to as **good quantum numbers**.
- All operators in the CSCO conmute with H:

$$[H, A] = [H, B] = [H, C] = ... = 0.$$

- They represent conserved quantities.

- **But in classical mechanics conserved quantities are a consequence of the existence of symmetries...**



Symmetries in Quantum Mechanics.

- In the quantum realm a classical transformation T is represented by an operator U_T in $\mathcal{H}$

$$|\psi\rangle \to |\psi'\rangle = U_T|\psi\rangle$$

- If T is a symmetry of the system, probabilities are invariant

$$|\langle\phi'|\psi'\rangle|^2 = |\langle U_T\phi|U_T\psi\rangle|^2 = |\langle\phi|U_T^\dagger U_T\psi\rangle|^2 = |\langle\phi|\psi\rangle|^2$$

- Symmetry transformations are represented by unitary (or anti-unitary, Wigner 1932) operators

$$U_T^\dagger U_T = U_T U_T^\dagger = \mathbf{1}$$

- Observables transform as

$$\mathcal{O}' = U_T \mathcal{O} U_T^\dagger$$

- The system is invariant thus

$$H' = U_T H U_T^\dagger = H \qquad \Rightarrow [H, U_T] = 0$$

- If the symmetry is continuous U_T can always be written in a exponential form (exponential map)

$$U_T = e^{iG} \qquad \text{with} \qquad G^\dagger = G.$$

- Continuous transformations depend on n parámetros $\alpha_i, \; i = 1, 2, ...n$. e.g. for $SO(3)$, $\alpha_i = \theta_i$.
- There are n generators G_i (e.g. for $SO(3)$, $G_i = J_i$). In general $G = G_i \alpha_i = \boldsymbol{G} \cdot \boldsymbol{\alpha}$.
- The generators of continuous symmetries are Hermitian operators. Represent observables in QM.
- $[U_T, H] = 0 \Rightarrow [G_i, H] = 0$. Generators of continuous symmetries are natural candidates for the CSCO.
- **Only those generators commuting with each other can be in the CSCO** : **Cartan Sub-algebra of the symmetry group**.
- In addition there are products of generators commuting with all G_i: **Casimir Operators of the symmetry group**. They also belong to the CSCO.

Conclusion: a quantum system is fully characterized by its total symmetry group

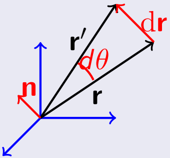
Good quantum numbers correspond to the eigenvalues of H, the Casimir operators of the full symmetry group and the generators in the Cartan sub-algebra of this group. Quantum states are classified into minimal subspaces invariant under the full symmetry group transformations (irreducible representaciones, **irreps** for short).

Elementary particles are quantum systems. We must find and classify all their symmetries:

1. Free particles: Space-time symmetries. Poincarè group: Rotations+translations+boosts. Supersymmetry?
2. Interacting particles: Gauge symmetries: $SU(3)_c \otimes SU(2)_L \otimes Y(1)_Y$.
3. Global symmetries: lepton number, baryon number etc.
4. Discrete symmetries: C, P, T, CP, PT, CPT, etc.

Starting: Rotations as a symmetry transformation in QM

Classical world



$$d\boldsymbol{r} = \boldsymbol{n} \times \boldsymbol{r}\, d\theta$$

$$x_i' = (\delta_{ik} - i(\boldsymbol{J}\cdot\boldsymbol{n})_{ik} d\theta) x_k$$

$$(J_i)_{jk} = -i\epsilon_{ijk}$$

$$[J_i, J_j] = i\epsilon_{ijk} J_k$$

$$[\boldsymbol{J}^2, J_i] = 0$$

$$R(\boldsymbol{n}, \theta) = e^{-i\boldsymbol{J}\cdot\boldsymbol{n}\theta}$$

Quantum realm

- $\boldsymbol{J} \equiv$ Rotation generators in $\mathcal{H}$
- Must satisfy $[J_i, J_j] = i\epsilon_{ijk} J_k$
- $R(\boldsymbol{n}, \theta) \to D_R(\boldsymbol{n}, \theta) \equiv e^{-i\boldsymbol{J}\cdot\boldsymbol{n}\theta}$
- If rotations are a symmetry : $[J_i, H] = 0 \Rightarrow [\boldsymbol{J}^2, H] = 0.$
- From the Lie Algebra $[\boldsymbol{J}^2, J_i] = 0$
- $\{H, \boldsymbol{J}^2, J_z\}$ are in the CSCO.
- Must calculate eigenstates and eigenvalues of $\{H, \boldsymbol{J}^2, J_z\}$.

Irreps of SU(2)

- Start with the Lie Algebra : $[J_i, J_j] = i\epsilon_{ijk} J_k$.
- Define the eigenstates of J^2 y J_z

$$J^2|a,b\rangle = a|a,b\rangle,$$
$$J_z|a,b\rangle = b|a,b\rangle.$$

- Define the operators $J_\pm \equiv J_x \pm iJ_y$

- Show that satisfy

$$J_-J_+ = \mathbf{J}^2 - J_z^2 - J_z,$$
$$J_+J_- = \mathbf{J}^2 - J_z^2 + J_z,$$
$$[J_z, J_\pm] = \pm J_\pm,$$
$$[J_+, J_-] = 2J_z,$$
$$[\mathbf{J}^2, J_\pm] = 0.$$

The following relations hold

$$\mathbf{J}^2 J_\pm|a,b\rangle = J_\pm \mathbf{J}^2|a,b\rangle = aJ_\pm|a,b\rangle,$$
$$J_zJ_\pm|a,b\rangle = ([J_z, J_\pm] + J_\pm J_z)|a,b\rangle = (\pm J_\pm + J_\pm b)|a,b\rangle = (b\pm 1)J_\pm|a,b\rangle$$
$$J_\pm|a,b\rangle \sim |a, b\pm 1\rangle$$

- Use the last relation k times

$$(J_\pm)^k |a,b\rangle \sim |a, b\pm k\rangle$$

- But

$$\mathbf{J}^2 - J_z^2 = \frac{1}{2}(J_+J_- + J_-J_+) = \frac{1}{2}\left(J_+(J_+)^\dagger + (J_+)^\dagger J_+\right)$$

thus

$$\begin{aligned}\langle a,b|\mathbf{J}^2 - J_z^2|a,b\rangle &= \frac{1}{2}(\langle a,b|J_+(J_+)^\dagger|a,b\rangle + \langle a,b|(J_+)^\dagger J_+|a,b\rangle) \\ &= \frac{1}{2}(||J_+^\dagger|a,b\rangle|| + ||J_+|a,b\rangle||) \geqslant 0\end{aligned}$$

hence

$$a - b^2 \geqslant 0 \Rightarrow b^2 \leq a.$$

- There must exist $b_{\text{máx}}$ y $b_{\text{mín}}$ such that

$$J_+|a, b_{\text{máx}}\rangle = 0 \qquad J_-|a, b_{\text{mín}}\rangle = 0$$

- From these relations

$$J_-J_+|a, b_{\text{máx}}\rangle = 0 \qquad \Rightarrow (\mathbf{J}^2 - J_z^2 - J_z)|a, b_{\text{máx}}\rangle = 0$$

$$a - b_{\text{máx}}^2 - b_{\text{máx}} = 0 \Rightarrow a = b_{\text{máx}}(b_{\text{máx}} + 1).$$

- Also

$$J_+J_-|a, b_{\text{mín}} >= (\mathbf{J}^2 - J_z^2 + J_z)|a, b_{\text{mín}}\rangle = 0$$

so

$$a = b_{\text{mín}}(b_{\text{mín}} - 1).$$

- Finally

$$b_{\text{máx}}(b_{\text{máx}} + 1) = b_{\text{mín}}(b_{\text{mín}} - 1) \qquad \Rightarrow b_{\text{máx}} = -b_{\text{mín}}.$$

- Furthermore, for some integer n

$$(J_+)^n|a, b_{\text{mín}}\rangle \sim |a, b_{\text{máx}}\rangle.$$

$$b_{\text{máx}} = b_{\text{mín}} + n \Rightarrow b_{\text{máx}} = \frac{n}{2} \Rightarrow a = \frac{n}{2}(\frac{n}{2} + 1)$$

- Define $j \equiv \frac{n}{2}$ such that $b_{\text{máx}} = j$ y $a = j(j+1)$. On the other side $b_{\text{mín}} = -j$ and $b_{\text{máx}} - b_{\text{mín}} = n$, hence $b = m$ where $m = -j, -j+1, ..., j-1, j$.
- In this notation

$$\mathbf{J}^2|j,m\rangle = j(j+1)|j,m\rangle, \qquad J_z|j,m\rangle = m|j,m\rangle$$

- As to the raising operator

$$J_+|jm\rangle = C_{jm}|j,m+1\rangle.$$

But

$$\langle jm|(J_+)^\dagger J_+|jm\rangle = |C_{jm}|^2 = \langle jm|\mathbf{J}^2 - J_z^2 - J_z|jm\rangle = [j(j+1) - m(m+1$$

- Assuming real coefficients (Condon-Shortley convention)

$$|C_{jm}| = \sqrt{j(j+1) - m(m+1)} = \sqrt{(j-m)(j+m+1)}$$

$$J_+|j,m\rangle = \sqrt{(j-m)(j+m+1)}|j,m+1\rangle$$

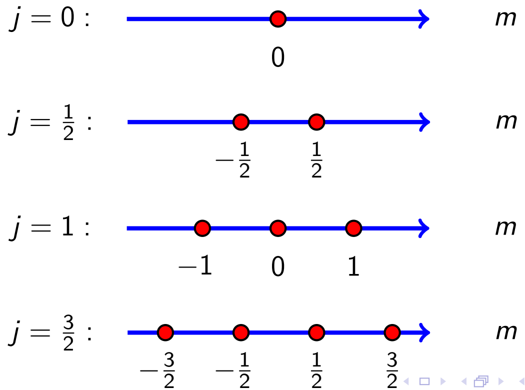
- Similarly

$$J_-|j,m\rangle = \sqrt{(j+m)(j-m+1)}|j,m-1\rangle$$

Summarizing: irreps de $SU(2)$

$$J^2|j,m\rangle = j(j+1)|j,m\rangle$$
$$J_z|j,m\rangle = m|j,m\rangle$$
$$J_\pm|j,m\rangle = \sqrt{(j\mp m)(j\pm m+1)}|j,m\pm 1\rangle$$

$$j=\frac{n}{2},$$
$$m=-j,-j+1,..,j.$$



Matrix representations

1. Notice
$$\langle |j', m'|D_R(\boldsymbol{n}, \theta)|jm\rangle \sim \delta_{j'j}$$
2. **Irreps are characterized by j**. Spanned by the set$\{|jm\rangle\}$, $m = -j, -j+1, ...j$.
3. These are orthogonal subspaces of dimension $2j + 1$.
4. Within these subspaces the generators have the following matrix representation
$$\begin{aligned}
\langle jm'|J_z|jm\rangle &= m\delta_{m'm} \\
\langle jm'|J_+|jm\rangle &= \sqrt{(j-m)(j+m+1)}\delta_{m',m+1} \\
\langle jm'|J_-|jm\rangle &= \sqrt{(j+m)(j-m+1)}\delta_{m',m-1}
\end{aligned}$$
5. These are $(2j + 1) \times (2j + 1)$matrices.

Defining representation: $j = \frac{1}{2}$

Shorthand notation: $|\frac{1}{2}, -\frac{1}{2}\rangle \equiv |-\rangle$, $|\frac{1}{2}, \frac{1}{2}\rangle \equiv |+\rangle$.

$$J_-|-\rangle = 0, \qquad J_-|+\rangle = |-\rangle, \qquad J_+|-\rangle = |+\rangle, \qquad J_+|+\rangle = 0.$$

Matrix representation for $j = \frac{1}{2}$

$$\langle jm'|J_z|jm\rangle = \begin{pmatrix} \langle +|J_z|+\rangle & \langle +|J_z|-\rangle \\ \langle -|J_z|+\rangle & \langle -|J_z|-\rangle \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \equiv \frac{1}{2}\sigma_z,$$

$$\langle jm'|J_+|jm\rangle = \begin{pmatrix} \langle +|J_+|+\rangle & \langle +|J_+|-\rangle \\ \langle -|J_+|+\rangle & \langle -|J_+|-\rangle \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \equiv \frac{1}{2}\sigma_+,$$

$$\langle jm'|J_-|jm\rangle = \begin{pmatrix} \langle +|J_-|+\rangle & \langle +|J_-|-\rangle \\ \langle -|J_-|+\rangle & \langle -|J_-|-\rangle \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \equiv \frac{1}{2}\sigma_-.$$

$$J_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \equiv \frac{1}{2}\sigma_x \qquad J_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \equiv \frac{1}{2}\sigma_y$$

Summarizing, for $j = \frac{1}{2}$

$$\boldsymbol{J} = \frac{1}{2}\boldsymbol{\sigma}$$

These matrices satisfy

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\ \sigma_k, \qquad \{\sigma_i, \sigma_j\} = 2\delta_{ij}.$$

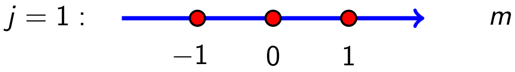
$$D^{(\frac{1}{2})}(\mathbf{n}, \theta) = \exp(-i\frac{\boldsymbol{\sigma}}{2} \cdot \boldsymbol{n}\theta) = \cos\frac{\theta}{2}\mathbf{1} - i\boldsymbol{\sigma} \cdot \boldsymbol{n}\ \sin\frac{\theta}{2}.$$

States are represented by two-component *spinors* in this basis

$$|+\rangle \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |-\rangle \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$D^{(\frac{1}{2})}(\mathbf{n}, \theta)$ are 2x2 unitary matrices of unit determinant: $\boldsymbol{SU}(2)$

Adjoint representation: $j = 1$



- A similar calculation yields

$$J_x = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \quad J_y = \begin{pmatrix} 0 & -\frac{i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & 0 \end{pmatrix}, \quad J_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

- These matrices are equivalent to $(\tilde{J}_i)_{jk} = -i\epsilon_{ijk}$

$$\tilde{J}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \tilde{J}_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad \tilde{J}_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

- Homework: Find the unitary matrix S connecting these representations: $\tilde{J}_i = SJ_iS^\dagger$.

The classical vectors (e.g. $\boldsymbol{r}$) transform in this representation

$SU(2)$ and the spin of elementary particles.

Modelo Estándar

(0,0)

Boson spin=0	Mass (GeV)
H Higgs	125.6

$$SU(3)_C \times SU(2)_L \times U(1)_Y$$

Leptons spin = 1/2		
Flavor	Mass GeV/c^2	Electric charge
ν_e electron neutrino	$<1\times10^{-8}$	0
e electron	0.000511	−1
ν_μ muon neutrino	<0.0002	0
μ muon	0.106	−1
ν_τ tau neutrino	<0.02	0
τ tau	1.7771	−1

Quarks spin = 1/2		
Flavor	Approx. Mass GeV/c^2	Electric charge
u up	0.003	2/3
d down	0.006	−1/3
c charm	1.3	2/3
s strange	0.1	−1/3
t top	175	2/3
b bottom	4.3	−1/3

(1/2,0), (0,1/2)

Strong (color) spin = 1		
Name	Mass GeV/c^2	Electric charge
g gluon	0	0

Unified Electroweak spin = 1		
Name	Mass GeV/c^2	Electric charge
γ photon	0	0
W^-	80.4	−1
W^+	80.4	+1
Z^0	91.187	0

(1/2,1/2)

$SU(2)$ and the spin of elementary particles.

- $j = 0$: Higgs $\to |0, 0\rangle$.
- $j = \frac{1}{2}$: quarks u, d, c, s, t, b and leptons e, μ, τ,

$$|e\rangle = \begin{cases} |\frac{1}{2}, \frac{1}{2}\rangle & \equiv |e \uparrow\rangle \\ |\frac{1}{2}, -\frac{1}{2}\rangle & \equiv |e \downarrow\rangle \end{cases}$$

- $j = 1$: gauge bosons: γ, W^+, W^-, Z^0, g.

$$|W\rangle = \begin{cases} |1, 1\rangle & \equiv |W \uparrow\rangle \\ |1, 0\rangle & \equiv |W \rightarrow\rangle \\ |1, -1\rangle & \equiv |W \downarrow\rangle \end{cases}$$

- Higher j: There are known composite particles transforming in these representations but not elementary particles, except for
- $j = 2$ graviton. Not a quantum theory so far. Not included in these lectures.
- We are familiar with the quantum description of NR spinless particles. How do we describe spinning particles?

Homogeneous Lorentz Group: Rotations + Boosts.

- HLG transformations in the classical world: Boosts along x

$$x' = \frac{x + vt}{1 - \frac{v^2}{c^2}}, \qquad y' = y, \qquad z' = z, \qquad t' = \frac{t + \frac{v}{c^2}x}{1 - \frac{v^2}{c^2}}.$$

- Define $\beta = \frac{v}{c}$, $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ $\quad x^1 = x,\ x^2 = y,\ x^3 = z,\ x^0 = ct.$
- Boost transformation along x^1 reads

$$x^{0'} = \gamma(x^0 + \beta x^1), \quad x^{1'} = \gamma(\beta x^0 + \beta x^1), \quad x^{2'} = x^2 \quad x^{3'} = x^3.$$

- The relation $\gamma^2 - \gamma^2\beta^2 = 1$ holds, we parametrize

$$\gamma = \cosh\varphi, \qquad \gamma\beta = \sinh\varphi.$$

- Finally

$$\begin{pmatrix} x^{0'} \\ x^{1'} \\ x^{2'} \\ x^{3'} \end{pmatrix} = \begin{pmatrix} \cosh\varphi & \sinh\varphi & 0 & 0 \\ \sinh\varphi & \cosh\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}.$$

- In Minkowski space a boost along x is done by the matrix

$$B_x(\varphi) = \begin{pmatrix} \cosh\varphi & \sinh\varphi & 0 & 0 \\ \sinh\varphi & \cosh\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- This is a continuous transformation. The corresponding generator is

$$K_x = i\frac{dB_x}{d\varphi}\mid_{\varphi=0} = i\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- Similarly for boosts along y and z we obtain

$$K_y = i\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad K_z = i\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

- Embedding rotations in Minkowski space we get

$$J_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad J_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad J_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- A general HLG transformation reads

$$x^{\mu'} = \Lambda^{\mu}{}_{\nu}(\boldsymbol{\theta}, \boldsymbol{\varphi}) x^{\nu}$$

- Boost generators satisfy

$$[K_i, K_j] = -i\epsilon_{ijk} J_k$$

Boosts do not form a group...but

$$\begin{aligned} [J_i, J_j] &= i\epsilon_{ijk} J_k \\ [J_i, K_j] &= i\epsilon_{ijk} K_k \\ [K_i, K_j] &= -i\epsilon_{ijk} J_k \end{aligned}$$

Boosts and rotations form a group: Homogeneous Lorentz Group. Rotations are a subgroup of the HLG.

Quantum realm: Irreps of the HLG.

- Define

$$\mathbf{A} = \frac{1}{2}(\mathbf{J} - i\mathbf{K}), \qquad \mathbf{B} = \frac{1}{2}(\mathbf{J} + i\mathbf{K}),$$

- These operators satisfy

$$\begin{aligned} [A_i, A_j] &= i\epsilon_{ijk} A_k \longrightarrow SU(2)_A \leftarrow Basis : \{|a, m_a\rangle\} \\ [B_i, B_j] &= i\epsilon_{ijk} B_k \longrightarrow SU(2)_B \leftarrow Basis : \{|b, m_b\rangle\} \\ [A_i, B_j] &= 0 \end{aligned}$$

- The HLG is isomorphic to : $SU(2)_A \otimes SU(2)_B$
- Basis of $\mathcal{H} = \{|a, m_a\rangle \otimes |b, m_b\rangle\}$.
- Casimir operators: $\{\boldsymbol{A}^2, \boldsymbol{B}^2\}$.
- Cartan subalgebra: $\{A_z, B_z\}$.
- Irreps of dimension $(2a+1)(2b+1)$ characterized by two quantum numbers (a, b).

$$\Lambda(\boldsymbol{\theta}, \boldsymbol{\varphi}) = e^{-i(\boldsymbol{J}\cdot\boldsymbol{\theta} + \boldsymbol{K}\cdot\boldsymbol{\varphi})} = e^{-i(\boldsymbol{A}\cdot(\boldsymbol{\theta}+i\boldsymbol{\varphi}) + \boldsymbol{B}\cdot(\boldsymbol{\theta}-i\boldsymbol{\varphi}))}$$

Irreps del HLG

$$(0,0)$$

$$(\tfrac{1}{2},0) \qquad (0,\tfrac{1}{2})$$

$$(1,0) \qquad (\tfrac{1}{2},\tfrac{1}{2}) \qquad (0,1)$$

$$(\tfrac{3}{2},0) \qquad (1,\tfrac{1}{2}) \qquad (\tfrac{1}{2},1) \qquad (0,\tfrac{3}{2})$$

$$(2,0) \qquad (\tfrac{3}{2},\tfrac{1}{2}) \qquad (1,1) \qquad (\tfrac{1}{2},\tfrac{3}{2}) \qquad (0,2)$$

Irreps del HLG

Higgs			$(0,0)$		
Quarks & Leptons		$(\frac{1}{2},0)$		$(0,\frac{1}{2})$	
Gauge bosons		$(1,0)$	$(\frac{1}{2},\frac{1}{2})$	$(0,1)$	
	$(\frac{3}{2},0)$	$(1,\frac{1}{2})$		$(\frac{1}{2},1)$	$(0,\frac{3}{2})$
Graviton	$(2,0)$	$(\frac{3}{2},\frac{1}{2})$	$(1,1)$	$(\frac{1}{2},\frac{3}{2})$	$(0,2)$

Chiral representations: $(a, 0)$ y $(0, b)$

If $a = 0$: $(0, b)$ representation

$$\mathbf{A} = 0 \quad \Rightarrow \quad \mathbf{J} = i\mathbf{K}$$

$$\Rightarrow \mathbf{B} = \frac{1}{2}(\mathbf{J} + i\mathbf{K}) = \mathbf{J}$$

If $b = 0$: $(a, 0)$ representation

$$\mathbf{B} = 0 \quad \Rightarrow \quad \mathbf{J} = -i\mathbf{K}$$

$$\Rightarrow \mathbf{A} = \frac{1}{2}(\mathbf{J} - i\mathbf{K}) = \mathbf{J}$$

$(0, b) = (0, j) \equiv$ **Left representation**

$$\Lambda_L(\boldsymbol{\theta}, \boldsymbol{\varphi}) = e^{-i\boldsymbol{J}\cdot(\boldsymbol{\theta} - i\boldsymbol{\varphi})}$$

$(a, 0) = (j, 0) \equiv$ **Right representation**

$$\Lambda_R(\boldsymbol{\theta}, \boldsymbol{\varphi}) = e^{-i\boldsymbol{J}\cdot(\boldsymbol{\theta} + i\boldsymbol{\varphi})}$$

Chiral representations transform identically under rotations but have opposite transformation properties under boosts.

- There are two representations with $j = \frac{1}{2}$.
- We cannot distinguish them in the non-relativistic limit.
- How is the electron in our atoms connected to these irreps?

Matrix representation of operators in the chiral irreps

- Rotations have the conventional $SU(2)$ matrix representations.
- Under a boost ($\boldsymbol{\theta} = 0$) in general:

$$|k^{\mu\prime}, j\rangle_R = B_R\left(k^\mu \to k^{\mu\prime}\right))|k^\mu, j\rangle_R = \exp(+\mathbf{J}\cdot\boldsymbol{\varphi})|k^\mu, j\rangle_R$$
$$|k^{\mu\prime}, j\rangle_L = B_L\left(k^\mu \to k^{\mu\prime}\right)|k^\mu, j\rangle_L = \exp(-\mathbf{J}\cdot\boldsymbol{\varphi})|k^\mu, j\rangle_L.$$

- For the particle rest frame $k^\mu = (m, \mathbf{0}) \to k^{\mu\prime} = (E, \boldsymbol{p})$:

$$\cosh\varphi = \gamma = \frac{E}{m}, \quad \sinh\varphi = \gamma\beta = \frac{E}{m}\frac{|\mathbf{p}|}{E} = \frac{|\mathbf{p}|}{m},$$
$$\cosh\frac{\varphi}{2} = \sqrt{\frac{E+m}{2m}}, \quad \sinh\frac{\varphi}{2} = \sqrt{\frac{E-m}{2m}}.$$

- For $j = \frac{1}{2}$ we get ($\boldsymbol{\varphi} = \boldsymbol{n}\varphi$):

$$B_R\left(\boldsymbol{\varphi}\right) = \exp(+\frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\varphi}) = \cosh\frac{\varphi}{2} + \boldsymbol{\sigma}\cdot\boldsymbol{n}\sinh\frac{\varphi}{2} = \frac{E+m+\boldsymbol{\sigma}\cdot\boldsymbol{p}}{\sqrt{2m(E+m)}}$$
$$B_L\left(\boldsymbol{\varphi}\right) = \exp(-\frac{\boldsymbol{\sigma}}{2}\cdot\boldsymbol{\varphi}) = \cosh\frac{\varphi}{2} - \boldsymbol{\sigma}\cdot\boldsymbol{n}\sinh\frac{\varphi}{2} = \frac{E+m-\boldsymbol{\sigma}\cdot\boldsymbol{p}}{\sqrt{2m(E+m)}}$$

- The matrix representation of the states $|\frac{1}{2}\frac{1}{2}\rangle_{R,L}$ and $|\frac{1}{2},-\frac{1}{2}\rangle_{R,L}$ in the rest frame are

$$
\begin{aligned}
\phi_L(\mathbf{0},+) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \phi_R(\mathbf{0},+) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\phi_L(\mathbf{0},-) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \phi_R(\mathbf{0},-) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\end{aligned}
$$

- Using the boost matrices we obtain

$$
\begin{aligned}
\phi_L(\mathbf{p},+) &= N\begin{pmatrix} E+m-p_z \\ -(p_x+ip_y) \end{pmatrix} \qquad \phi_R(\mathbf{p},+) = N\begin{pmatrix} E+m+p_z \\ p_x+ip_y \end{pmatrix} \\
\phi_L(\mathbf{p},-) &= N\begin{pmatrix} -(p_x-ip_y) \\ E+m+p_z \end{pmatrix} \qquad \phi_R(\mathbf{p},-) = N\begin{pmatrix} p_x-ip_y \\ E+m-p_z \end{pmatrix}
\end{aligned}
$$

with $N = \frac{1}{\sqrt{2m(E+m)}}$

Parity and irreps of the HLG

- Under parity

$$\boldsymbol{J} \to \boldsymbol{J}, \quad \boldsymbol{K} \to -\boldsymbol{K} \Rightarrow \boldsymbol{A} \to \boldsymbol{B}, \quad \mathbf{B} \to \boldsymbol{A} \Rightarrow (a,b) \leftrightarrow (b,a)$$

- If $a \neq b$, the subspace (a,b) is not an irrep for parity. Need to consider $(a,b) \oplus (b,a)$.
- In particular, for $(j,0) \oplus (0,j)$

$$\Lambda(\boldsymbol{\theta},\boldsymbol{\varphi}) = \begin{pmatrix} \Lambda_R(\boldsymbol{\theta},\boldsymbol{\varphi}) & 0 \\ 0 & \Lambda_L(\boldsymbol{\theta},\boldsymbol{\varphi}) \end{pmatrix}, \quad \omega(\boldsymbol{p},\lambda) = \begin{pmatrix} \phi_R(\boldsymbol{p},\lambda) \\ \phi_L(\boldsymbol{p},\lambda) \end{pmatrix}.$$

- In the rest frame, parity has the matrix representation

$$\Pi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- In this frame, states with well defined parity (π) satisfy

$$\Pi\omega(\mathbf{0},\lambda) = \pi\, \omega(\mathbf{0},\lambda), \qquad \pi = \pm 1$$

- Boosting this equation

$$B(\boldsymbol{p})[\Pi - \pi]B^{-1}(\boldsymbol{p})B(\boldsymbol{p})\omega(\mathbf{0}) = 0 \Rightarrow [B(\boldsymbol{p})\Pi B^{-1}(\boldsymbol{p}) - \pi]\omega(\boldsymbol{p}) = 0.$$

- But under boosts

$$\Pi \boldsymbol{K} \Pi = -\boldsymbol{K} \qquad \Rightarrow \qquad \Pi\ B(\boldsymbol{p})\Pi = B^{-1}(\boldsymbol{p}),$$

- The boosted parity eigenvalue equation reads

$$[B^2(\boldsymbol{p})\Pi - \pi]\omega(\boldsymbol{p}) = 0.$$

- Explicitly

$$\begin{pmatrix} -\pi & \exp(2\mathbf{J}\cdot\mathbf{n}\varphi) \\ \exp(-2\mathbf{J}\cdot\mathbf{n}\varphi) & -\pi \end{pmatrix} \omega(\boldsymbol{p}, \lambda) = 0$$

- For $j = \frac{1}{2}$

$$\exp(\pm 2\frac{\boldsymbol{\sigma}}{2}\cdot\mathbf{n}\varphi) = \cosh\varphi \pm \boldsymbol{\sigma}\cdot\mathbf{n}\sinh\varphi = \frac{E \pm \boldsymbol{\sigma}\cdot\boldsymbol{p}}{m}.$$

- For $j = \frac{1}{2}$ the parity condition in an arbitrary frame reads

$$\begin{pmatrix} -\pi & \frac{E+\boldsymbol{\sigma}\cdot\boldsymbol{p}}{m} \\ \frac{E-\boldsymbol{\sigma}\cdot\boldsymbol{p}}{m} & -\pi \end{pmatrix} \omega(\boldsymbol{p}, \lambda) = 0.$$

- Defining the following matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

- The parity condition reads

$$[\gamma^\mu p_\mu - \pi m]\omega(\boldsymbol{p}, \lambda, \pi) = 0.$$

- Dirac equation is just the covariant form of parity eigenvalue equation in $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$.
- The γ^μ matrices satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}.$$

- The explicit form of these matrices change if we change the chosen basis for $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$.

Configuration space: Dirac Equation

- Take positive parity $\pi = 1$ and define the configuration space "wave function"

$$\psi(x) = \omega(\boldsymbol{p}, \lambda, \pi) e^{-ip.x}, \qquad i\partial^\mu \psi(x) = p^\mu \psi(x).$$

- It satisfies Dirac equation

$$[i\gamma^\mu \partial_\mu - m]\, \psi(x) = 0.$$

- "Probability conservation" : $\partial^\mu J_\mu(\mathbf{r}, t) = 0$, with

$$J^\mu(\mathbf{r}, t) = \bar{\psi}\gamma^\mu \psi, \qquad \bar{\psi} \equiv \psi^\dagger \gamma^0$$

- "Probability density"

$$\rho(\mathbf{r}, t) = \bar{\psi}\gamma^0 \psi = \psi^\dagger \psi$$

HLG irreps, quirality and elementary particles

- $j = 0$: Higgs $\rightarrow (0, 0)$. Is not chiral.
- $j = \frac{1}{2}$: quarks u, d, c, s, t, b and leptons e, μ, τ, ν_i,

$$|f, \lambda\rangle \rightarrow \begin{cases} |f, \lambda\rangle_R & \rightarrow (\frac{1}{2}, 0) \\ |f, \lambda\rangle_L & \rightarrow (0, \frac{1}{2}) \end{cases}$$

  - Electron in atoms has positive parity (convention)

$$|e^-, \lambda\rangle = \frac{1}{\sqrt{2}}(|e, \lambda\rangle_R + |e, \lambda\rangle_L).$$

  - The orthogonal state has the opposite parity: antiparticle.

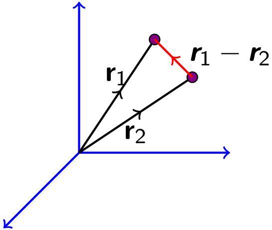
$$|e^+, \lambda\rangle = \frac{1}{\sqrt{2}}(|e, \lambda\rangle_R + |e, \lambda\rangle_L).$$

- $j = 1$: gauge bosons : γ, W^+, W^-, Z^0, *gluon*. transform in the $(\frac{1}{2}, \frac{1}{2})$ irrep. Not chiral.

Warming up: Why do we need fields?

Because physical laws are local

- Coulomb interaction: $\boldsymbol{F}_{12} = \frac{q_1 q_2}{r_{12}^2}\boldsymbol{e}_{12}$.
- Action at a distance.
- Change: $\boldsymbol{F}_1 = q_1 \boldsymbol{E}_2$, $\boldsymbol{E}_2(\boldsymbol{r}_1) = \frac{q_2}{r_{12}^2}\boldsymbol{e}_{12}$.
- Interaction is related to the value of fields at a given space point $\boldsymbol{r}$.
- Fields are produced by **sources**. In general they also depend on t.
- A particle in an e.m. field feels the Lorentz Force:



$$\boldsymbol{F}(\boldsymbol{r}, t) = q\left(\boldsymbol{E}(\boldsymbol{r}, t) + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}(\boldsymbol{r}, t)\right).$$

Why do we need quantum fields?

- From $E^2 = P^2c^2 + m^2c^4$: Klein Gordon equation for a free particle

$$\hbar^2 \frac{\partial^2 \phi(\mathbf{r}, t)}{\partial t^2} = \left[-\hbar^2 c^2 \boldsymbol{\nabla}^2 + m^2 c^4\right] \phi(\mathbf{r}, t),$$

or in covariant form ($x^0 = ct$)

$$\left[\partial^\mu \partial_\mu + \frac{m^2c^2}{\hbar^2}\right] \phi(\mathbf{r}, t) = 0,$$

- "Probability conservation" : $\partial^\mu J_\mu(\mathbf{r}, t) = 0$, with

$$J^\mu(\mathbf{r}, t) = \frac{i}{\hbar} \left[\phi^* \partial^\mu \phi - (\partial^\mu \phi^*)\phi\right].$$

- "Probability density"

$$\rho(\mathbf{r}, t) = \frac{i}{\hbar} \left[\phi^* \frac{\partial \phi}{\partial t} - \frac{\partial \phi^*}{\partial t} \phi\right].$$

- Free particle solutions : $\phi(x) = Ne^{-\frac{i}{\hbar}p\cdot x}$ yield

$$E = \pm\sqrt{\boldsymbol{p}^2 c^2 + m^2 c^4} \Rightarrow \rho = |N|^2 2E$$

- **Negative energy solutions and negative probabilities**. No way.
- Coupling to an electromagnetic field A^μ: Lorentz force $\Rightarrow$ "minimal coupling" : $\hat{P}^\mu \to \hat{P}^\mu - \frac{q}{c}A^\mu$

$$\left[(\partial^\mu + i\frac{q}{\hbar c}A^\mu)(\partial_\mu + i\frac{q}{\hbar c}A_\mu) + \frac{m^2c^2}{\hbar^2}\right]\phi(\mathbf{r},t) = 0.$$

- Complex conjugated equation

$$\left[(\partial^\mu - i\frac{q}{\hbar c}A^\mu)(\partial_\mu - i\frac{q}{\hbar c}A_\mu) + \frac{m^2c^2}{\hbar^2}\right]\phi^*(\mathbf{r},t) = 0.$$

Notice:

- If ϕ is a solution for a particle of charge q and mass m then ϕ^* is a solution for a particle of charge $-q$ and same mass ("anti-particle").
- These solutions cannot be equal (describe particles with the opposite charge). Both are solutions of the free KG equation with mass m.
- Conclusion: KG equation describes multiparticle states, at least a pair of particles with same mass and opposite charge.

Single particle relativistic quantum mechanics is not tenable.

Multiparticle RQM= Quantum Fields

Dirac equation and charge conjugation

- Consider a Dirac particle in an external e.m. field

$$[i\gamma^\mu(\partial_\mu - \frac{ie}{c}A_\mu) - m]\psi(x) = 0.$$

- The conjugate field $\psi^c = i\gamma^2\psi^*$ satisfy

$$[i\gamma^\mu(\partial_\mu {\color{red}+} \frac{ie}{c}A_\mu) - m]\psi^c(x) = 0.$$

- ψ^c describes a state with the same mass and spin but opposite quantum numbers $\boldsymbol{p}, \pi, \lambda$ and electric charge: antiparticle.
- Conclusion: Dirac equation describes multiparticle states, at least a pair of particles.

Single particle relativistic quantum mechanics is not tenable.

Multiparticle RQM= Quantum Field Theory

Natural units

- Fundamental constants and units:

$$[\hbar] = Et, \quad [c] = L/t, \quad [\hbar c] = EL, \quad [mc^2] = E, \quad [k_B T] = E$$

- Time, lenght, mass and temperature

$$[\frac{\hbar}{E}] = t, \quad [\frac{\hbar c}{E}] = L, \quad [\frac{E}{c^2}] = m, \quad [\frac{E}{k_B}] = T$$

- Fundamental constants apart we get

$$t = [\frac{1}{E}], \quad L = [\frac{1}{E}], \quad m = [E], \quad T = [E]$$

Basic conversion factors:

$$1seg = 1{,}5 \times 10^{24} GeV^{-1} \hbar \qquad H_0 = 1{,}53 \times 10^{-42} GeV/\hbar$$

$$1m = 5 \times 10^{15} GeV^{-1} \hbar c \qquad G_F = 1{,}16 \times 10^{-5} GeV^{-2} (\hbar c)^3$$

$$1Kg = 5{,}62 \times 10^{26} GeV/c^2 \qquad M_P = 1{,}22 \times 10^{19} GeV/c^2$$

From point particles to classical fields

Point particles:

- Lagrangian and Action

$$L = L(q_r, \dot{q}_r, t), \qquad S = \int L(q_r, \dot{q}_r, t) dt$$

- Equation of motion

$$\delta S = 0 \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) - \frac{\partial L}{\partial q_r} = 0$$

Classical fields:

- $t \to x^\mu$, $q_r \to \phi_r(x^\mu)$, $\quad L(\phi_r, \partial^\mu \phi_r) = \int \mathcal{L}(\phi_r, \partial^\mu \phi_r) d^3x$
- Equation of motion: $S = \int \mathcal{L}(\phi_r, \partial^\mu \phi_r) d^4x$

$$\delta S = 0 \quad \Rightarrow \quad \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi_r)}\right) - \frac{\partial \mathcal{L}}{\partial \phi_r} = 0.$$

Klein-Gordon Lagrangian: complex scalar field

$$\mathcal{L} = \partial^\mu \phi^* \partial_\mu \phi - m^2 \phi^* \phi + \Omega$$

$$\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi^*)} = \partial_\mu \phi, \qquad \frac{\partial \mathcal{L}}{\partial \phi^*} = -m^2 \phi$$

$$\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} = \partial_\mu \phi^*, \qquad \frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi^*$$

Euler-Lagrange equations for ϕ^* and ϕ yield

$$(\partial^\mu \partial_\mu + m^2)\phi = 0$$

$$(\partial^\mu \partial_\mu + m^2)\phi^* = 0$$

Conserved currents

Noether's theorem

Every continuous symmetry of the action yields a conserved current $\partial^\mu J_\mu = 0$.

Consider the transformation of the fields: $\delta\phi_r \equiv X_r(\phi)$. If $\delta S = 0$ then the Lagrangian density can change at most by a total divergence $\delta\mathcal{L} = \partial^\mu F_\mu(\phi)$. But

$$\begin{aligned}\delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\phi_r}\delta\phi_r + \frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi_r)}\delta(\partial^\mu\phi_r) \\ &= (\frac{\partial\mathcal{L}}{\partial\phi_r} - \partial^\mu\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi_r)})\delta\phi_r + \partial^\mu(\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi_r)}\delta\phi_r).\end{aligned}$$

Thus

$$\partial^\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi_r)}X_r(\phi) - F_\mu(\phi)\right) = 0.$$

Conserved charges

Current conservation

$$\frac{\partial J^0}{\partial t} = -\boldsymbol{\nabla} \cdot \boldsymbol{J}$$

Integrating in a given volume

$$\int_V d^3x \frac{\partial J^0}{\partial t} = \frac{d}{dt}\int_V d^3x J^0 = -\int_V d^3x \boldsymbol{\nabla} \cdot \boldsymbol{J} = -\int_S \boldsymbol{J} \cdot d\boldsymbol{a}$$

thus the charge

$$Q \equiv \int d^3x J^0$$

is a locally conserved quantity.

Space-time translations

- Space-time translations: $x^\mu \to x^\mu + \epsilon^\mu$
- $\phi_r \to \phi_r + \epsilon^\mu \partial_\mu \phi_r \Rightarrow \delta\phi_r = \epsilon^\mu \partial_\mu \phi_r = X_r$.
- $\mathcal{L} \to \mathcal{L} + \epsilon^\mu \partial_\mu \mathcal{L} \Rightarrow \delta\mathcal{L} = \partial_\mu(\epsilon^\mu \mathcal{L}) \Rightarrow F^\mu = \epsilon^\mu \mathcal{L}$.
- Noether's current

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi_r)} \epsilon^\alpha \partial_\alpha \phi_r - \epsilon^\mu \mathcal{L}$$

- There is a conserved current for each value of ϵ^ν

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi_r)} \partial_\nu \phi_r - g^\mu{}_\nu \mathcal{L}, \qquad \partial_\mu T^\mu{}_\nu = 0.$$

- Conserved charges: energy and momentum

$$E = \int d^3x T^0{}_0 = \int d^3x \left(\frac{\partial \mathcal{L}}{\partial(\partial^0 \phi_r)} \partial_0 \phi_r - \mathcal{L} \right),$$

$$P_i = \int d^3x T^0{}_i = \int d^3x \frac{\partial \mathcal{L}}{\partial(\partial^0 \phi_r)} \partial_i \phi_r.$$

For the KG field

$$\mathcal{L} = \partial^\mu \phi^* \partial_\mu \phi - m^2 \phi^* \phi + \Omega$$

$$E = \int d^3x [\partial^0 \phi^* \partial_0 \phi + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi - \Omega],$$
$$P_i = \int d^3x [\partial^0 \phi^* \partial_i \phi + \partial_i \phi^* \partial^0 \phi].$$

Internal symmetries: $U(1)$

The KG Lagrangian is invariant under global $U(1)$ transformations

$$\phi \to \phi' = U(\theta)\phi \equiv e^{-iq\theta}\phi \Rightarrow \delta\phi = -iq\theta\phi, \quad \delta\phi^* = iq\theta\phi^*$$

The Noether current is

$$J^\mu = iq\left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi^*)}\phi^* - \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)}\phi\right)$$
$$= iq(\phi^* \partial^\mu \phi - (\partial^\mu \phi^*)\phi)$$

Hamiltonian formalism

- Calculate the momentum density

$$\pi_r(x) \equiv \frac{\partial \mathcal{L}}{\partial(\partial^0 \phi_r)}.$$

- Define the Hamiltonian density as

$$\mathcal{H}(\phi_r, \pi_r) = \pi_r \dot{\phi}_r - \mathcal{L}.$$

where $\mathcal{L}$ and $\dot{\phi}_r$ are written in terms of π_r and ϕ_r.

- Hamilton equations

$$\dot{\phi}_r = \frac{\partial \mathcal{H}}{\partial \pi_r}, \qquad \dot{\pi}_r = -\frac{\partial \mathcal{H}}{\partial \phi_r}$$

- For the KG field (Excercise 1)

$$\mathcal{H}(\phi, \phi^\dagger, \pi, \pi^\dagger) = \pi^\dagger \pi + \nabla \phi^\dagger \cdot \nabla \phi + m^2 \phi^\dagger \phi - \Omega.$$

Notice this is a positive difinite quantity!

Quantization of a complex scalar field

The classical field satisfies KG equations whose solutions are

$$\phi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_{\boldsymbol{p}}}}[a_{\boldsymbol{p}} e^{i\ p.x} + b^\dagger_{\boldsymbol{p}} e^{-i\ p.x}]$$

$$\phi^\dagger(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_{\boldsymbol{p}}}}[a^\dagger_{\boldsymbol{p}} e^{-i\ p.x} + b_{\boldsymbol{p}} e^{i\ p.x}],$$

with $E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$. The corresponding momenta are

$$\pi(x) \equiv \frac{\partial \mathcal{L}}{\partial(\partial^0 \phi)} = \dot{\phi}^\dagger(x), \qquad \pi^\dagger(x) \equiv \frac{\partial \mathcal{L}}{\partial(\partial^0 \phi^\dagger)} = \dot{\phi}(x).$$

Quantization

Consider ϕ and $\phi^\dagger$ as operators $\Rightarrow$ promote $a_{\boldsymbol{p}}$ and $b_{\boldsymbol{p}}$ to operators.

In NRQM, the Heisenberg picture operators $X_i(t)$, $P_j(t)$ satisfy

$$[X_i(t), P_j(t)] = i\hbar\delta_{ij}, \qquad [X_i(t), X_j(t)] = 0, \qquad [P_i(t), P_j(t)] = 0.$$

Field quantization is realized imposing the *equal time commutation relations*

$$[\phi_i(\boldsymbol{x}, t), \pi_j(\boldsymbol{y}, t)] = i\delta_{ij}\boldsymbol{\delta}(\boldsymbol{x} - \boldsymbol{y}),$$
$$[\phi_i(\boldsymbol{x}, t), \phi_j(\boldsymbol{y}, t)] = 0, \qquad [\pi_i(\boldsymbol{x}, t), \pi_j(\boldsymbol{y}, t)] = 0.$$

These commutators require

$$[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}] = \delta^3(\boldsymbol{p} - \boldsymbol{q}), \quad [a_{\boldsymbol{p}}, a_{\boldsymbol{q}}] = 0, \quad [a^\dagger_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}] = 0,$$
$$[b_{\boldsymbol{p}}, b^\dagger_{\boldsymbol{q}}] = \delta^3(\boldsymbol{p} - \boldsymbol{q}), \quad [b_{\boldsymbol{p}}, b_{\boldsymbol{q}}] = 0, \quad [b^\dagger_{\boldsymbol{p}}, b^\dagger_{\boldsymbol{q}}] = 0,$$
$$[a_{\boldsymbol{p}}, b_{\boldsymbol{q}}] = 0, \quad [a_{\boldsymbol{p}}, b^\dagger_{\boldsymbol{q}}] = 0, \quad [a^\dagger_{\boldsymbol{p}}, b_{\boldsymbol{q}}] = 0, \quad [a^\dagger_{\boldsymbol{p}}, b^\dagger_{\boldsymbol{q}}] = 0.$$

Homework

- Write

$$\phi(x) = \int d^3p[a_{\boldsymbol{p}} f_p^*(x) + b_{\boldsymbol{p}}^\dagger f_p(x)], \qquad f_p(x) \equiv \frac{e^{-i\ p.x}}{\sqrt{(2\pi)^3 2E_{\boldsymbol{p}}}}$$

- Show that $(F \overleftrightarrow{\partial^0} G \equiv F\partial^0 G - (\partial^0 F)G)$

$$\langle f_p | f_q \rangle \equiv \int d^3x f_p^*(x) i \overleftrightarrow{\partial^0} f_q(x) = \delta^3(\boldsymbol{q} - \boldsymbol{p}).$$

- Use this relation to show that

$$a_{\boldsymbol{q}} = \int d^3x \phi(x) i \overleftrightarrow{\partial^0} f_q(x), \quad a_{\boldsymbol{q}}^\dagger = \int d^3x f_q^*(x) i \overleftrightarrow{\partial^0} \phi^\dagger(x)$$

$$b_{\boldsymbol{q}} = \int d^3x \phi^\dagger(x) i \overleftrightarrow{\partial^0} f_q(x), \quad b_{\boldsymbol{q}}^\dagger = \int d^3x f_q^*(x) i \overleftrightarrow{\partial^0} \phi(x)$$

- Use these results to calculate the commutators in the previous slide.

Noether's charges

A straightforward calculation yields

$$H \equiv \int [\dot{\phi}^\dagger \dot{\phi} + \nabla \phi^\dagger \cdot \nabla \phi + m^2 \phi^\dagger \phi - \Omega] d^3x$$

$$= \int d^3p \; E_{\boldsymbol{p}} \left(a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}} + \frac{1}{2}\delta^3(\mathbf{0}) + b^\dagger_{\boldsymbol{p}} b_{\boldsymbol{p}} + \frac{1}{2}\delta^3(\mathbf{0}) \right) - \Omega V$$

$$\boldsymbol{P} = \int d^3x [\dot{\phi}^\dagger \nabla \phi + (\nabla \phi^\dagger) \dot{\phi}]$$

$$= \int d^3p \; \boldsymbol{p} \left(a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}} + \frac{1}{2} + b^\dagger_{\boldsymbol{p}} b_{\boldsymbol{p}} + \frac{1}{2} \right),$$

$$Q = q \int d^3x \phi^\dagger i \overleftrightarrow{\partial^0} \phi = q \int d^3p \left(a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}} - b^\dagger_{\boldsymbol{p}} b_{\boldsymbol{p}} \right),$$

These operators commute with each other

$$[H, \boldsymbol{P}] = 0, \qquad [H, Q] = 0, \qquad [Q, \boldsymbol{P}] = 0.$$

- Defining $N_p^a \equiv a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}}$ and $N_p^b = b_{\boldsymbol{p}}^\dagger b_{\boldsymbol{p}}$ we get

$$[N_p^a, N_q^b] = 0, \qquad [N_p^a, N_q^a] = 0, \qquad [N_p^b, N_q^b] = 0$$
$$[N_p^a, a_{\boldsymbol{q}}^\dagger] = a_{\boldsymbol{p}}^\dagger \delta^3(\boldsymbol{p}-\boldsymbol{q}), \quad [N_p^a, a_{\boldsymbol{q}}] = -a_{\boldsymbol{p}} \delta^3(\boldsymbol{p}-\boldsymbol{q}),$$
$$[N_p^b, b_{\boldsymbol{q}}^\dagger] = b_{\boldsymbol{p}}^\dagger \delta^3(\boldsymbol{p}-\boldsymbol{q}), \quad [N_p^b, b_{\boldsymbol{q}}] = -b_{\boldsymbol{p}} \delta^3(\boldsymbol{p}-\boldsymbol{q}).$$

- Define now the total N operators

$$N^a = \int d^3p \; a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}}, \qquad N^b = \int d^3p \; b_{\boldsymbol{p}}^\dagger b_{\boldsymbol{p}}.$$

- Notice that $Q = q(N^a - N^b)$. These operators satisfy

$$[N^a, N^b] = 0, \quad [N^{a,b}, H] = 0, \quad [N^{a,b}, \boldsymbol{P}] = 0, \quad [N^{a,b}, Q] = 0.$$

The operators $\{H, \boldsymbol{P}, Q, N^a, N^b\}$ form a complete set of commuting operators

Quantum states are labelled by their eigenvalues (good quantum numbers): $|E, \boldsymbol{p}, q(n^a - n^b), n^a, n^b\rangle$.

Harmonic oscillator in NRQM: Brief review

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = (a^\dagger a + \frac{1}{2})\hbar\omega, \quad a \equiv \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right)$$

Ladder operators satisfy $[a, a^\dagger] = 1$. Define the number operator as

$$N \equiv a^\dagger a \quad \Rightarrow \quad [N, a^\dagger] = a^\dagger, \quad [N, a] = -a.$$

Eigenstates of the number operator

$$N|n\rangle = n|n\rangle \quad \Rightarrow \quad Na^\dagger|n\rangle = (n+1)a^\dagger|n\rangle, \quad Na|n\rangle = (n-1)a|n\rangle$$

Hamiltonian positive difinite $\Rightarrow$ $n = 0, 1, 2, \ldots$

$$H|n\rangle = (n + \frac{1}{2})\hbar\omega|n\rangle \qquad |n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle, \quad a|0\rangle = 0$$

The ground state $|0\rangle$ has nonvanishing energy $E = \hbar\omega/2$.

Interpretation

- The KG field consists of a collection of HO's, two (a or b) for each value of $\boldsymbol{p}$.
- The ground state of the $\boldsymbol{p}$-HO satisfy

$$N_p^a|0_p\rangle_a = 0, \qquad N_p^b|0_p\rangle_b = 0.$$

- Denote the collective ground state by $|0\rangle = \Pi_{\boldsymbol{p}}^{\otimes}|0_p\rangle$
- For the ground state we get

$$N^a|0\rangle = \int d^3p\ N_{\boldsymbol{p}}^a|0\rangle = 0, \qquad N^b|0\rangle = \int d^3p\ N_{\boldsymbol{p}}^b|0\rangle = 0$$

$$H|0\rangle = \left[\int d^3p\ E_{\boldsymbol{p}}(N_{\boldsymbol{p}}^a + N_{\boldsymbol{p}}^b + \delta^3(\mathbf{0})) - \Omega V\right]|0\rangle \equiv (\mathcal{E}_0 - \Omega)V|0\rangle,$$

$$\boldsymbol{P}|0\rangle = \int d^3p\ \boldsymbol{p}(N_{\boldsymbol{p}}^a + N_{\boldsymbol{p}}^b)|0\rangle = 0,$$

$$Q|0\rangle = q\int d^3p(N_p^a - N_p^b)|0\rangle = 0.$$

Energy of the ground state

- The zero-point energy of the harmonic oscillators yield

$$\int d^3p\ E_{\boldsymbol{p}}\delta^3(\mathbf{0}) = \int d^3p\ E_{\boldsymbol{p}} \int \frac{d^3x}{(2\pi)^3} e^{i\boldsymbol{q}.\boldsymbol{x}}|_{\boldsymbol{q}=0} = \frac{V}{(2\pi)^3} \int d^3p\ E_{\boldsymbol{p}}$$

- This is formally an infinite amount of energy.
- However, we have learnt that our theories are valid only up to some energy (effective theories).
- We expect QFT description to be appropriate up to some energy scale $\Lambda \gg m$. The corresponding energy density is

$$\mathcal{E}_0 = \int \frac{d^3p}{(2\pi)^3}\ E_{\boldsymbol{p}} = \frac{4\pi}{(2\pi)^3} \int_m^\Lambda \sqrt{E^2 - m^2} E^2 dE = \frac{\Lambda^4}{8\pi^2}.$$

- Choose $\Omega = \mathcal{E}_0$

$$H = \int d^3p\ E_{\boldsymbol{p}}(N^a_{\boldsymbol{p}} + N^b_{\boldsymbol{p}}), \qquad H|0\rangle = 0.$$

First excited states: sectors $(n_a, n_b) = (1,0), (0,1)$

- There are two types of first excited states

$$a^\dagger_{\boldsymbol{p}}|0\rangle, \qquad b^\dagger_{\boldsymbol{q}}|0\rangle.$$

- Acting on them with the H and P we get

$$Ha^\dagger_{\boldsymbol{q}}|0\rangle = \int d^3p\ E_{\boldsymbol{p}} N^a_p a^\dagger_{\boldsymbol{q}}|0\rangle = \int d^3p\ E_{\boldsymbol{p}}[N^a_p, a^\dagger_{\boldsymbol{q}}]|0\rangle = E_{\boldsymbol{q}} a^\dagger_{\boldsymbol{q}}|0\rangle,$$

$$Hb^\dagger_{\boldsymbol{q}}|0\rangle = \int d^3p\ E_{\boldsymbol{p}} N^b_p b^\dagger_{\boldsymbol{q}}|0\rangle = \int d^3p\ E_{\boldsymbol{p}}[N^b_p, b^\dagger_{\boldsymbol{q}}]|0\rangle = E_{\boldsymbol{q}} b^\dagger_{\boldsymbol{q}}|0\rangle,$$

$$\boldsymbol{P}a^\dagger_{\boldsymbol{q}}|0\rangle = \int d^3p\ \boldsymbol{p} N^a_p a^\dagger_{\boldsymbol{q}}|0\rangle = \int d^3p\ \boldsymbol{p}[N^a_p, a^\dagger_{\boldsymbol{q}}]|0\rangle = \boldsymbol{q} a^\dagger_{\boldsymbol{q}}|0\rangle,$$

$$\boldsymbol{P}b^\dagger_{\boldsymbol{q}}|0\rangle = \int d^3p\ \boldsymbol{p} N^b_p b^\dagger_{\boldsymbol{q}}|0\rangle = \int d^3p\ \boldsymbol{p}[N^b_p, b^\dagger_{\boldsymbol{q}}]|0\rangle = \boldsymbol{q} b^\dagger_{\boldsymbol{q}}|0\rangle, \ldots$$

- These states are eigenstates of the H and P with the same energy and momentum (hence same mass).

- But they have opposite $U(1)$ charge eigenvalues...

$$Qa^\dagger_{\boldsymbol{q}}|0\rangle = q\int d^3pN^a_pa^\dagger_{\boldsymbol{q}}|0\rangle = q\int d^3p\,[N^a_p, a^\dagger_{\boldsymbol{q}}]|0\rangle = qa^\dagger_{\boldsymbol{q}}|0\rangle,$$
$$Qb^\dagger_{\boldsymbol{q}}|0\rangle = -q\int d^3pN^b_pb^\dagger_{\boldsymbol{q}}|0\rangle = -q\int d^3p\,[N^b_p, b^\dagger_{\boldsymbol{q}}]|0\rangle = -qb^\dagger_{\boldsymbol{q}}|0\rangle.$$

- ...and different N^a, N^b eigenvalues

$$N^aa^\dagger_{\boldsymbol{q}}|0\rangle = \int d^3pN^a_pa^\dagger_{\boldsymbol{q}}|0\rangle = \int d^3p\,[N^a_p, a^\dagger_{\boldsymbol{q}}]|0\rangle = a^\dagger_{\boldsymbol{q}}|0\rangle,$$
$$N^ab^\dagger_{\boldsymbol{q}}|0\rangle = 0, \qquad N^ba^\dagger_{\boldsymbol{q}}|0\rangle = 0,$$
$$N^bb^\dagger_{\boldsymbol{q}}|0\rangle = \int d^3pN^b_pb^\dagger_{\boldsymbol{q}}|0\rangle = \int d^3p\,[N^b_p, b^\dagger_{\boldsymbol{q}}]|0\rangle = b^\dagger_{\boldsymbol{q}}|0\rangle.$$

- Finally

$$a^\dagger_{\boldsymbol{p}}|0\rangle = |E, \boldsymbol{p}, q, 1, 0\rangle \equiv |\boldsymbol{p}\rangle_a, \qquad b^\dagger_{\boldsymbol{p}}|0\rangle = |E, \boldsymbol{p}, -q, 0, 1\rangle \equiv |\boldsymbol{p}\rangle_b$$

The state $|\boldsymbol{p}\rangle_a \equiv a^\dagger_{\boldsymbol{p}}|0\rangle$ describes a "quantum" (particle) with momentum $\boldsymbol{p}$, energy $E_{\boldsymbol{p}}$ mass m and $U(1)$ charge q.

The state $|\boldsymbol{p}\rangle_b \equiv b^\dagger_{\boldsymbol{p}}|0\rangle$ describes a "quantum" with momentum $\boldsymbol{p}$, energy $E_{\boldsymbol{p}}$ mass m and $U(1)$ charge $-q$: anti-particle!!!

- Normalization of single particle states

$$\langle \boldsymbol{p}|\boldsymbol{q}\rangle = \langle 0|a_{\boldsymbol{p}} a^\dagger_{\boldsymbol{q}}|0\rangle = \langle 0|[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}]|0\rangle = \delta^3(\boldsymbol{p}-\boldsymbol{q}).$$

- Vacuum-to-single-particle transition amplitude induced by the field

$$\langle \boldsymbol{q}|\phi^\dagger(x)|0\rangle = \langle \boldsymbol{q}| \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_{\boldsymbol{p}}}} [a^\dagger_{\boldsymbol{p}} e^{-i\ p.x} + b_{\boldsymbol{p}} e^{i\ p.x}]|0\rangle$$
$$= \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_{\boldsymbol{p}}}} \langle \boldsymbol{q}|a^\dagger_{\boldsymbol{p}}|0\rangle e^{-i\ p.x} = \frac{e^{-i\ q.x}}{\sqrt{(2\pi)^3 2E_{\boldsymbol{q}}}}.$$

Sector with two particles: $(n^a, n^b) = (2,0), (1,1), (0,2)$

- Sucesively acting with two creation operators two-particle excited states:

$$|\boldsymbol{p}_1, \boldsymbol{p}_2\rangle_{aa} = a^\dagger_{\boldsymbol{p}_1} a^\dagger_{\boldsymbol{p}_2}|0\rangle, \quad |\boldsymbol{p}_1, \boldsymbol{p}_2\rangle_{ab} = a^\dagger_{\boldsymbol{p}_1} b^\dagger_{\boldsymbol{p}_2}|0\rangle$$
$$|\boldsymbol{p}_1, \boldsymbol{p}_2\rangle_{ba} = b^\dagger_{\boldsymbol{p}_1} a^\dagger_{\boldsymbol{p}_2}|0\rangle, \quad |\boldsymbol{p}_1, \boldsymbol{p}_2\rangle_{bb} = b^\dagger_{\boldsymbol{p}_1} b^\dagger_{\boldsymbol{p}_2}|0\rangle,$$

- $[a^\dagger_{\boldsymbol{p}_i}, a^\dagger_{\boldsymbol{p}_j}] = 0, \quad [b^\dagger_{\boldsymbol{p}_i}, b^\dagger_{\boldsymbol{p}_j}] = 0 \quad \Rightarrow$ Bose statistics.
- The $N^{a,b}$ operators count the total number of each type of quanta e.g.

$$\begin{aligned} N^a a^\dagger_{\boldsymbol{p}_1} a^\dagger_{\boldsymbol{p}_2}|0\rangle &= \int d^3p \; N^a_p a^\dagger_{\boldsymbol{p}_1} a^\dagger_{\boldsymbol{p}_2}|0\rangle \\ &= \int d^3p \; [\delta^3(\boldsymbol{p} - \boldsymbol{p}_1) a^\dagger_{\boldsymbol{p}} + a^\dagger_{\boldsymbol{p}_1} N^a_p] a^\dagger_{\boldsymbol{p}_2}|0\rangle \\ &= 2 a^\dagger_{\boldsymbol{p}_1} a^\dagger_{\boldsymbol{p}_2}|0\rangle \end{aligned}$$

- Energy and momentum are additive quantum numbers, e.g.

$$
\begin{aligned}
Ha^{\dagger}_{\boldsymbol{p}_1} a^{\dagger}_{\boldsymbol{p}_2}|0\rangle &= \int d^3p E_{\boldsymbol{p}}\ N^a_p a^{\dagger}_{\boldsymbol{p}_1} a^{\dagger}_{\boldsymbol{p}_2}|0\rangle \\
&= \int d^3p\ E_{\boldsymbol{p}}[\delta^3(\boldsymbol{p}-\boldsymbol{p}_1)a^{\dagger}_{\boldsymbol{p}} + a^{\dagger}_{\boldsymbol{p}_1} N^a_p]a^{\dagger}_{\boldsymbol{p}_2}|0\rangle \\
&= (E_{\boldsymbol{p}_1} + E_{\boldsymbol{p}_2})a^{\dagger}_{\boldsymbol{p}_1} a^{\dagger}_{\boldsymbol{p}_2}|0\rangle
\end{aligned}
$$

- Vacuum-to-two-particle-states transition is done by the product of two fields e.g. ($d\tilde{p} \equiv d^3p,\ f_p(x) \equiv \frac{e^{ip.x}}{\sqrt{(2\pi)^3 2E}}$)

$$
\begin{aligned}
\phi^{\dagger}(x)\phi(y)|0\rangle &= \int d\tilde{p}d\tilde{k}[a^{\dagger}_{\boldsymbol{p}} f^*_p(x) + b_{\boldsymbol{p}} f_p(x)][a_{\boldsymbol{k}} f_k(y) + b^{\dagger}_{\boldsymbol{k}} f^*_k(y)]|0\rangle \\
&= \int d\tilde{p}d\tilde{k}[f^*_p(x) f^*_k(y) a^{\dagger}_{\boldsymbol{p}} b^{\dagger}_{\boldsymbol{k}}|0\rangle + \int d\tilde{p} f_p(x) f^*_p(y)|0\rangle
\end{aligned}
$$

- Products of n fields acting on the vacuum produces n-particle states like $|\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3 ...\rangle_a = a^{\dagger}_{\boldsymbol{p}_1} a^{\dagger}_{\boldsymbol{p}_2} a^{\dagger}_{\boldsymbol{p}_3 ...}|0\rangle$
- $[a^{\dagger}_{\boldsymbol{p}_i}, a^{\dagger}_{\boldsymbol{p}_j}] = 0 \quad \Rightarrow$ scalar particles obey Bose statistics.

Summary

1. Elementary particles are quantum systems.
2. As such, they are characterized by the good quantum numbers corresponding to the eigenvalues of the Hamiltonian, the Casimir Operators and the operators in the Cartan sub-algebra of the full symmetry group.
3. For free particles the obvious symmetries are are rotations, boosts and space-time translations.
4. We obtained the irreps of rotations $SU(2)$: Spin.
5. We worked out the irreps of the HLG and HLG+Parity: Spin, Quirality and Dirac equation.
6. The full symmetry group is the Poincarè group. No time for this, but the Casimir operators yield two quantum numbers: mass and spin.
7. Single particle RQM is not tenable.
8. Quantum Field Theory allows us to follow this construction. Quantum theory of multi-particle states.