Electromagnetism

Remember

\[ \vec{E} = -\nabla \phi - \dot{\vec{A}} \quad \quad \vec{B} = \nabla \wedge \vec{A} \]

Maxwell equations

\[ \nabla \cdot \vec{E} = \rho, \quad \nabla \wedge \vec{B} = \dot{\vec{E}} + \vec{J} \]

\[ \nabla \cdot \vec{B} = 0, \quad \nabla \wedge \vec{E} = -\dot{\vec{B}} \]

Conservation

\[ \dot{\rho} = \nabla \cdot \vec{J} \]
Electromagnetism

Can collect E and B in a tensor given by

\[ F_{\mu\nu} = \begin{pmatrix}
0 & -E_x & E_y & E_z \\
E_x & 0 & B_x & -B_y \\
E_y & -B_x & 0 & B_x \\
E_z & B_y & -B_z & 0
\end{pmatrix} \]

And the charge density and current in 4-vector

\[ j^\mu = (\rho, \vec{J}) \]

Can be constructed from

\[ A^\mu = (\phi, \vec{A}) \quad \Rightarrow \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \]
Electromagnetism

Maxwell equations are

\[ \nabla \cdot \vec{E} = \rho, \quad \nabla \wedge \vec{B} = \vec{E} + \vec{J} \]

\[ \nabla \cdot \vec{B} = 0, \quad \nabla \wedge \vec{E} = -\vec{B} \]

Equations valid in Minkowski, but it is straightforward to generalise them to curved space-time, just \( \partial_\mu \rightarrow \nabla_\mu \)

\[ \partial_\mu F^{\mu \nu} = 4\pi j^\nu \]

\[ \partial_{[\mu} F_{\alpha \beta]} = 0 \]

\[ \partial_\mu F_{\alpha \beta} + \partial_\alpha F_{\beta \mu} + \partial_\beta F_{\mu \alpha} = 0 \]
Electromagnetism

- Energy density
  \[ W = \frac{1}{8\pi} (E^2 + B^2) \]

- Poynting vector
  \[ \vec{S} = \frac{1}{8\pi} \vec{E} \wedge \vec{B} \]

- Maxwell stress tensor
  \[ t_{\alpha\beta} = \frac{1}{4\pi} \left( \frac{1}{2} (E^2 + B^2) \delta_{\alpha\beta} - E_\alpha E_{\beta\alpha} - B_\alpha B_{\beta\alpha} \right) \]

- They satisfy
  \[ \nabla_\mu F^{\mu\nu} = 4\pi j^{\nu} \]
  \[ \nabla_{[\mu} F_{\nu\beta]} = 0 \]
Perfect Fluids

• A popular choice of matter in GR is a perfect fluid, defined to be a fluid with **NO viscosity** or **heat flow** and **isotropic** in its rest frame!

It is completely specified by its energy density and isotropic pressure

\[ T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu} \]

where \( U^\mu \) is the fluid's 4-velocity. If \( U^\mu U^\nu g_{\mu\nu} = -1 \)

\[ g_{\mu\nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \]

• Bianchi identity ( \( \nabla_\mu G^{\mu\nu} = 0 \) ) implies local energy-momentum conservation

\[ \nabla_\mu T^{\mu\nu} = 0 \]
Einstein Equations

\[ G_{\mu\nu} = 8\pi G_N T_{\mu\nu} \]

**Geometría del Espacio-tiempo**

**Materia/Energía**
El espacio-tiempo es como una “manta” invisible deformado por la materia o energía.
Einstein Equations

Cosmological Constant

\[ G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu} \]

Its divergence is zero.

Must be small otherwise we would have seen it in lab experiments!
Einstein Equations

Lovelock Theorem

If we want second order differential field equations for the metric with a 4d space-time, the only form is

\[ \alpha \left( G_{\mu\nu} + \Lambda g_{\mu\nu} \right) = 8\pi G_N T_{\mu\nu} \]

To recover Newtonian physics \( \alpha = 1 \)
Einstein Equations

Cauchy Problem. e.g. Maxwell eqns.

No $t$ derivatives \( \nabla \cdot E = 0, \quad \nabla \cdot B = 0 \) Constraints

1st order $t$ der. \( \dot{E} = \nabla \times B, \quad \dot{B} = -\nabla \times E \) Evolution eqns.

Give suitable initial data so that the system has only one solution. We need \( E(0,x), B(0,x) \) which satisfy the constraints, then \( E(t,x), B(t,x) \) are obtained from the evolution equations.
Einstein Equations

Cauchy Problem. Similar for RG

1) \( G_{00} \) No \( t \) derivatives Constraints
2) \( G_{0i} \) 1st order \( t \) der. Constraints
3) \( G_{ij} \) 2nd order \( t \) der. Evolution eqns.

Find \( g_{\mu\nu}, g_{\mu\nu,0} \equiv \partial_0 g_{\mu\nu} \) for \( x^0 = 0 \), which satisfy the constraints initially, and use the evolution equations to solve for \( x^0 > 0 \)

*The equations are linear in second derivatives!*
Einstein Equations

Relevant components of the metric

\( g_{\mu \nu} \) has 10 independent components (remember it is symmetric), but there are 6 evolution equations to determine them (?).

Actually, we have an arbitrary choice of coordinates. Therefore, there are only 10-4=6 variables to determine, which can be found using the evolution equations.
Variational Principles

Action?

Need to understand integration on a manifold

(will not look at exterior calculus, but just the basics)
Tensorial densities

Important objects in physics which are not tensors

1) volume factor \( d^4 x = dx^0 \, dx^1 \, dx^2 \, dx^3 \)

Under \( x^\mu \rightarrow x'^\mu \) transforms as

\[
d^4 x \rightarrow \det \left( \frac{\partial x^\mu}{\partial x'^\nu} \right) d^4 x
\]
Tensorial densities

Important objects in physics which are not tensors

2) determinants

\[ g \equiv \det g_{\mu\nu} \]

Transforms as

\[ g \rightarrow \left[ \det \left( \frac{\partial x^\mu}{\partial x'^\nu} \right) \right]^{-2} g \]

1) and 2) are call **tensor densities** because transform as some powers of the Jacobian
Tensorial densities

Tensor density of weight $w$

$$\tilde{T}^\hat{a}\_\hat{b} = J^w P^\hat{a}\_a P^\hat{b}\_b \tilde{T}^a\_b$$

where $J \equiv \det P^\hat{a}\_b = \det \left( \frac{\partial x^\hat{a}}{\partial x^b} \right)$ is the Jacobian!

The covariant derivative of a tensor density is arbitrary and not a tensor density in general.
Tensorial densities

Important objects in physics which are not tensors

Notice

\[ \int f(x^\mu) \sqrt{-g} d^4x \]

Is invariant under a change of coordinate, since \( f(x) \) is a scalar
Tensorial densities

Important objects in physics which are not tensors

Notice

\[ \int f(x^\mu) \sqrt{-g} d^4x \]

Is invariant under a change of coordinate, since \( f(x) \) is a scalar

Therefore the right way of writing integrals is using

\[ \int d^4x \rightarrow \int \sqrt{-g} d^4x \]
Hilbert-Einstein action

Einstein equations can be obtained from

\[ S = \int \sqrt{-g} d^4x \left( \frac{1}{16\pi G_N} R + \mathcal{L}_m \right) \]

\[ \frac{\delta S}{\delta g_{\mu\nu}} = 0 \]

\[ \frac{1}{8\pi G_N} \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g_{\mu\nu}} \equiv T_{\mu\nu} \]
Hilbert-Einstein action

Can include a Cosmological Constant

\[ S = \int \sqrt{-g} d^4x \left( \frac{1}{16\pi G_N} (R - 2\Lambda) + \mathcal{L}_m \right) \]

\[ \frac{\delta S}{\delta g_{\mu\nu}} = 0 \quad \Rightarrow \quad G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu} \]
Hilbert-Einstein action

Matter Lagrangians (examples)

1) Scalar field

\[ \mathcal{L}_m = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \]

\[ T_{\mu\nu} = -\partial_\mu \phi \partial_\nu \phi + g_{\mu\nu} \mathcal{L}_m \]
Hilbert-Einstein action

Matter Lagrangians (examples)

2) Electromagnetism

\[ \mathcal{L}_m = -\frac{1}{4g} g^{\mu\nu} g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} \]

\[ T_{\mu\nu} \propto F_{\mu\alpha} F^{\alpha}_{\nu} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \]
Alternative approaches

À la Palatini

Consider the connection and the metric as independent

\[ S = \int \sqrt{-g} d^4x \left( \frac{1}{16\pi G_N} (R - 2\Lambda) + \mathcal{L}_m \right) \]

\[ R(\Gamma, g) \]

\[ \frac{\delta S}{\delta g_{\mu\nu}} = 0 \quad \Rightarrow \quad G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu} \]

\[ \frac{\delta S}{\delta \Gamma^\sigma_{\mu\nu}} = 0 \quad \Rightarrow \quad \nabla^\sigma g_{\mu\nu} = 0 \]

The metric connection
3+1 Formalism

Also known as Hamiltonian or ADM

Consider the space-time splitting

\[ g_{\mu\nu} = \begin{pmatrix} -N^2 + \gamma_{ij} N^i N^j & \gamma_{ik} N^k \\ \gamma_{jk} N^k & \gamma_{ij} \end{pmatrix} \]

- \( N \) lapse function
- \( N^i \) shift function
- \( g_{ij} \) spatial metric

Schematically
3+1 Formalism

2) Hamiltonian (ADM or 3+1 formalism)

Extrinsic curvature

\[ K_{ij} = \frac{1}{2N} \left[ \partial_t (\gamma_{ij}) - \nabla_i N_j - \nabla_j N_i \right] \]

Then one gets the Hamiltonian

\[ S = \frac{1}{16\pi G_N} \int \sqrt{-g} d^4x \left( \partial_t \gamma_{ij} \Pi^{ij} - N H - N_i H^i \right) \]

Hamiltonian

\[ H = -\sqrt{\gamma} (^{(3)}R) + \frac{1}{\sqrt{\gamma}} \left( K_{ij} K^{ij} - (K^i_i)^2 \right) \]

Momentum

\[ H^i = 2\nabla_j \Pi^{ij} \]
3+1 Formalism

2) Hamiltonian (ADM or 3+1 formalism)

Observations

- Only $\gamma_{ij}$ is dynamical
- $N, N^i$ are Lagrange multipliers which lead to (in vacuum)

\[
H \simeq 0, \quad H^i \simeq 0
\]

Dynamical D.O.F.

\[6 - (1+3) = 2 \quad \Rightarrow \quad \text{two polarization modes of the graviton}\]
Weak field

Linearised gravity

Gravitational waves
Linearised gravity

Informal way
Use

\[ g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu} \]

and do an expansion in \( \epsilon \ll 1 \)

But does not make any sense formally, since we are comparing tensors (two metrics) on different manifolds!
Linearised gravity

Formal way

Take a 5d spacetime, with a foliation of 4d hypersurfaces that are connected by a congruence (map).
Linearised gravity

Formal way

Gauges (choice of map in the 5d setting – choice of coordinates from 4d!)

If

\[ x^\mu \rightarrow x^\mu + \epsilon \xi^\mu \]

Then

\[ h_{\mu\nu} \rightarrow h_{\mu\nu} + 2\epsilon \xi_{(\mu,\nu)} \]

Only gauge invariant quantities are physically relevant!
Linearised gravity

\[ g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu} \]

Inverse metric is

\[ g^{\mu\nu} = \eta^{\mu\nu} - \epsilon h^{\mu\nu} + \mathcal{O}(\epsilon) \]

And the Christoffel symbols

\[ \Gamma_{\nu\sigma}^{\mu} = \frac{1}{2} \epsilon \eta^{\mu\delta} \left[ h_{\delta\nu,\sigma} + h_{\delta\sigma,\nu} - h_{\sigma\nu,\delta} \right] \]
Hence

\[ R \sim \partial \Gamma - \partial \Gamma + \Gamma^2 - \Gamma^2 \]

Is invariant under the change of gauge

\[ h_{\mu \nu} \rightarrow h_{\mu \nu} + 2\epsilon \xi_{(\mu, \nu)} \]
It is useful to define the trace-reversed metric perturbation

\[ \bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h \]

where \( h \) is the trace of \( h_{\mu\nu} \)

The inverse is

\[ h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \bar{h} \]
Einstein's field equations

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N T_{\mu\nu} \]

become

\[ \bar{h}_{\mu\nu,\sigma}^\sigma + \eta_{\mu\nu} \bar{h}_{\sigma\delta}^{,\sigma\delta} - \bar{h}_{\mu\sigma,\nu}^\sigma - \bar{h}_{\nu\sigma,\mu}^\sigma = 16\pi G_N \epsilon^{-1} T_{\mu\nu} \]
Linearised gravity

Donder or harmonic gauge

Under a coord. transformation \( x^\mu \rightarrow x^\mu + \epsilon \xi^\mu \)

one gets \( \bar{h}_{\mu\nu}^{\ ,\nu} \rightarrow \bar{h}_{\mu\nu}^{\ ,\nu} + \Box \xi_\mu \)

So one can always choose \( \bar{h}_{\mu\nu}^{\ ,\nu} = 0 \)
Linearised gravity

The Einstein equations reduce to

$$\Box \bar{h}_{\mu\nu} = 16\pi G_N \epsilon^{-1} T_{\mu\nu}$$

(Together with the gauge condition $\bar{h}_{\mu\nu}^{;\nu} = 0$)
Gravitational radiation

In vacuum, the linearised equations are

\[ \Box \bar{h}_{\mu\nu} = 0 \quad \bar{h}_{\mu\nu,\nu} = 0 \]

Which can be solved using a single Fourier mode

\[ \bar{h}_{\mu\nu} = \hat{h}_{\mu\nu}(n_{\sigma})e^{in_{\sigma}x^{\sigma}} \]
Gravitational radiation

Then, these two equations

\[ \Box \bar{h}_{\mu\nu} = 0 \quad \bar{h}_{\mu\nu,\nu} = 0 \]

reduce to the conditions

\[ n_\sigma n^\sigma \hat{h}_{\mu\nu} = 0 \]

\[ n^\sigma \hat{h}_{\mu\sigma} = 0 \]
Gravitational radiation

Then, these two equations

\[ \square \bar{h}_{\mu\nu} = 0 \quad \bar{h}_{\mu\nu}^{\nu} = 0 \]

reduce to the conditions

\[ n_\sigma n^\sigma \hat{h}_{\mu\nu} = 0 \quad \text{In particular:} \quad n_\sigma = w(-1, 0, 0, 1) \]

\[ n^\sigma \hat{h}_{\mu\sigma} = 0 \]
Gravitational radiation

There is still some gauge freedom left, which can be used to set

\[ \tilde{h}_{0\nu} = 0 \quad \tilde{h} = 0 \]

\[ h_{\mu\nu} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & h_+ & h_x & 0 \\
0 & h_x & -h_+ & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \]

The two polarization modes of a gravitational wave!
Exact Solutions

Maximally symmetric

Black Holes
Maximally symmetric solns

Consider vacuum with a cosmological constant

\[ G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \]

There are 3 maximally symmetric solutions

- Minkowski \( \Lambda = 0, \quad R = 0 \)
- De Sitter (dS) \( \Lambda > 0, \quad R = \frac{2}{3} \Lambda \)
- Anti-de Sitter (AdS) \( \Lambda < 0, \quad R = \frac{2}{3} \Lambda \)
Detour: Penrose diagrams

- Causal and topological structure
- Infinite is brought to a finite distance
- Light rays propagate at 45°
- Conformal factors $g_{\mu\nu} \rightarrow \Omega(x^\alpha)g_{\mu\nu}$ do not change the lightcone structure, so can be dropped
- Useful to study black holes
Consider Minkowski spacetime

\[ ds^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2 \quad -\infty < t < \infty, \quad 0 < r < \infty \]

Change variables to

\[ u_\pm = \frac{1}{2} (t \pm r) \quad -\infty < u_+ < \infty, \quad -\infty < u_- < \infty \]

\[ u_- < u_+ \]

\[ ds^2 = -4du_+ du_- + (u_+ - u_-)^2 d\Omega_2^2 \]
Detour: Penrose diagrams

\[ ds^2 = -4dU_+dU_- + (u_+ - u_-)^2 d\Omega_2^2 \]

Bring infinite to fine distance with

\[ U_\pm = \tan^{-1}(u_\pm) \quad -\frac{\pi}{2} < U_+ < \frac{\pi}{2}, \quad -\frac{\pi}{2} < U_- < \frac{\pi}{2} \]

\[ U_- < U_+ \]

\[ ds^2 = \frac{1}{\cos^2(U_+) \cos^2(U_-)} \left( -4dU_+dU_- + \sin^2(U_+ - U_-)d\Omega_2^2 \right) \]
Detour: Penrose diagrams

\[
\begin{align*}
\frac{ds^2}{1} &= \frac{1}{\cos^2(U_+) \cos^2(U_-)} \left( -4dU_+dU_- + \sin^2(U_+ - U_-)d\Omega^2_2 \right) \\
& \text{Back to the Cartesian coords.} \\
\eta &= U_+ + U_-, \quad \chi = U_+ - U_- \\
-\pi < \eta < \pi, \quad -\pi < \chi < \pi \\
\frac{ds^2}{1} &= \Omega^{-2}(\eta, \chi) \left( -d\eta^2 + d\chi^2 + \sin^2(\chi)d\Omega^2_2 \right) \\
\Omega &= \frac{1}{2} (\cos(\eta) + \cos(\chi))
\end{align*}
\]
Einstein Static Universe (ESU)
Detour: Penrose diagrams

Einstein Static Universe (ESU)

\[ ds^2 = -d\eta^2 + d\chi^2 + \sin^2(\chi)d\Omega_2^2 \]

Solution of Einstein equations with a perfect fluid with

\[ \rho = \frac{3}{8\pi G_N} \]
\[ p = -\frac{1}{8\pi G_N} \]
\[ T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu} \]

Or a cosmological constant with a particular value!

Minkowski ↔ ESU
Detour: Penrose diagrams

\[ ds^2 = -d\eta^2 + d\chi^2 + \sin^2(\chi)d\Omega_2^2 \]

Half of 3-sphere

\(-\pi < \eta < \pi, \quad -\pi < \chi < \pi \)
Detour: Penrose diagrams

Take half of triangle and attach a full 3-sphere at each point
Detour: Penrose diagrams

- Future timelike infinity
- Future null infinity
- Spatial infinity
- Past null infinity
- Past timelike infinity
Detour 2: Horizons

Particle horizon

Event horizon

\[ x^\mu(\lambda) \]
Minkowski Penrose diagram
de Sitter (dS)

\[ R_{\rho\sigma\mu\nu} = \frac{\Lambda}{3} (g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}) \]

\[ \Lambda > 0, \quad R = \frac{2}{3} \Lambda \]

Easier to picture in 5d

\[ ds_{5d}^2 = -(dx^0)^2 + \sum_{i=1}^{4} (dx^i)^2 \]

\[ -(x^0)^2 + \sum_{i=1}^{4} (x^i)^2 = \frac{\Lambda}{3} \]
de Sitter (dS)

Geodesics

Planes through the origin

1) Hyperbola -> Timelike
2) Straight Line -> Null
3) Ellipse -> Spacelike
de Sitter (dS)

By choosing $x^A$, ($A = 0, \ldots, 4$) one can write the metric in the following useful frames:

**Static coordinates**

$$ds^2 = - \left(1 - \frac{\Lambda}{3} r^2\right) dt^2 + \left(1 - \frac{\Lambda}{3} r^2\right)^{-1} dr^2 + r^2 d\Omega_2^2$$

$$x^0 = \sqrt{\Lambda^2 - 9 \sinh(t)/3}$$

$$x^4 = \sqrt{\Lambda^2 - 9 \cosh(t)/3}$$

$$x^i = r \hat{x}^i, \quad i = 1, 2, 3 \quad \sum_{i} (\hat{x}^i)^2 = 1$$
de Sitter (dS)

Or using cosmological coordinates (steady-state Universe)

\[ ds^2 = -dt^2 + \exp(2Ht) \left( dr^2 + r^2 d\Omega_2^2 \right) \]

\[ H^2 = \frac{\Lambda}{3} \]

\[ x^0 = \sinh(t)/H + H r^2 e^{Ht}/2 \]
\[ x^4 = \sinh(t)/H - H r^2 e^{Ht}/2 \]
\[ x^i = r e^{Ht} \hat{x}^i, \quad i = 1, 2, 3 \]
\[ \sum_i (\hat{x}^i)^2 = 1 \]

Only covers half of hyperboloid

Exponential expansion!

Bondi, Gold, Hoyle
de Sitter (dS)

Penrose diagram

Both timelike and null future infinity
de Sitter (dS)

Penrose diagram
de Sitter (dS)

Penrose diagram for expanding Universe (SSU)
Recipes: Anti de Sitter (AdS)

Maximal symmetry

\[ R_{\rho\sigma\mu\nu} = \frac{\Lambda}{3} (g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}) \]

\[ \Lambda < 0, \quad R = \frac{2}{3}\Lambda \]

5d picture

\[ ds_{5d}^2 = -(dx^0)^2 - (dx^4)^2 + \sum_{i=1}^{3} (dx^i)^2 \]

Two times

\[ -(x^0)^2 - (x^4)^2 + \sum_{i=1}^{3} (x^i)^2 = -\frac{|\Lambda|}{3} \]
Recipes: Anti de Sitter (AdS)

Closed Timelike Curves (travel back in time?)

Choose the covering space (unwrap the circle to \( \mathbb{R} \)) and the closed timelike curves disappear (relevant to physics!)
Recipes: Anti de Sitter (AdS)

By choosing $x^A$, $(A = 0, \ldots, 4)$ one can write the metric as

**Static coordinates**

$$ds^2 = - \left(1 + \frac{|\Lambda|}{3} r^2\right) dt^2 + \left(1 + \frac{|\Lambda|}{3} r^2\right)^{-1} dr^2 + r^2 d\Omega_2^2$$

**Penrose diagram:** cannot be drawn well...

**Relevant for AdS/CFT**
Solutions so far...

<table>
<thead>
<tr>
<th>Solution</th>
<th>$\Lambda$</th>
<th>$T_{\mu\nu}$</th>
<th>Symmetry</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minkowski</td>
<td>0</td>
<td>0</td>
<td>Maximal</td>
</tr>
<tr>
<td>de Sitter</td>
<td>$&gt;0$</td>
<td>0</td>
<td>Maximal</td>
</tr>
<tr>
<td>Anti de Sitter</td>
<td>$&lt;0$</td>
<td>0</td>
<td>Maximal</td>
</tr>
<tr>
<td>Schwarzschild</td>
<td>0</td>
<td>$M\delta(r)$</td>
<td>Spherical</td>
</tr>
</tbody>
</table>
Birkhoff's theorem

If one assumes spherical symmetry in GR in vacuum

\[ ds^2 = -f_0(t, r)dt^2 + f_1(t, r)dt\,dr + f_2(t, r)dr^2 + f_3(t, r)d\Omega_2^2 \]

(\textit{using the gauge freedom can fix } f_1 = 0, \ f_3 = r^2), \textit{ then}

- The solution is unique
- Static (i.e. \( f_i \) do not depend on time)
- Characterised by one integration constant alone (\( M \))

\textit{It is called the Schwarzschild solution}
Schwarzschild solution

In detail

Choose

\[ ds^2 = -e^{2\alpha(t,r)} \, dt^2 + e^{2\beta(t,r)} \, dr^2 + r^2 \, d\Omega_2^2 \]

\[ R_{00} = \left[ \ddot{\beta} + (\dot{\beta})^2 - \dot{\alpha} \dot{\beta} \right] + e^{2(\alpha - \beta)} \left[ \alpha'' + (\alpha')^2 - \alpha' \beta' + 2\alpha' / r \right] \]

\[ R_{11} = - \left[ \alpha'' + (\alpha')^2 - \alpha' \beta' + 2\beta' / r \right] + e^{-2(\alpha - \beta)} \left[ \ddot{\beta} + (\dot{\beta})^2 - \dot{\alpha} \dot{\beta} \right] \]

\[ R_{01} = 2 \frac{\dot{\beta}}{r} \]

\[ R_{22} = e^{-2\beta} \left[ r(\beta' - \alpha') - 1 \right] + 1 \]

\[ R_{33} = \sin^2 \theta R_{22} \]

\[ \dot{'} = \partial_r, \quad \cdot = \partial_t \]

In vacuum, we have \( R_{\mu\nu} = 0 \)
Schwarzschild solution

Therefore

\[ R_{01} = 2 \frac{\dot{\beta}}{r} = 0, \quad \dot{\beta} = \beta(r) \]

\[ \partial_t \left[ R_{22} = e^{-2\beta} \left[ r(\beta' - \alpha') - 1 \right] + 1 \right] \quad \dot{\alpha}' = 0 \]

\[ \alpha(t, r) = f(r) + g(t) \]

The metric is

\[ ds^2 = -e^{2f(r)} e^{2g(t)} dt^2 + \ldots \]

By changing time, can always make \( g(t) = 0 \)
Schwarzschild solution

Now take the combination

\[ 0 = e^{2(\beta-\alpha)} R_{00} + R_{11} = \frac{2}{r}(\alpha' + \beta') \]

\[ \alpha = -\beta + c \]

*Can always choose the constant to be zero.*

**Now**

\[ R_{22} = 0 \quad \Rightarrow \quad (r e^{2\alpha})' = 1 \]

\[ e^\alpha = 1 + \frac{\mu}{r} \]
Schwarzschild solution

Since it is unique, it describes from black holes to planets!

If $r \gg 1$

$$g_{11} = \left(1 - \frac{\mu}{r}\right)^{-1} \approx 1 - \frac{\mu}{r} \sim 1 - 2\Phi = 1 + \frac{2G_N M}{r}$$

$$\mu = -\frac{2G_N M}{r}$$
Schwarzschild solution

Schwarzschild solution is then

\[ ds^2 = - \left( 1 - \frac{2G_N M}{r} \right) dt^2 + \left( 1 - \frac{2G_N M}{r} \right)^{-1} dr^2 + r^2 d\Omega_2^2 \]
Schwarzschild solution

Test particle in the $\theta = \pi/2$ plane

\[ \frac{1}{2} \left( \frac{dr}{d\tau} \right)^2 + V(r) = \frac{1}{2} E^2 \]

Particle of unit mass and energy $E$ in 1d potential

Kinetic potential energy conservation

For the Schwarzschild metric the potential is

\[ \frac{1}{2} \epsilon - \epsilon \frac{G N M}{r} + \frac{L^2}{2r^2} - \frac{G N M L^2}{r^3} \]

Const. Newtonian potential angular momentum

Only GR correction (Mercury)
Schwarzschild solution

Re-write $V(r)$ as

$$V(r) = \frac{1}{2r^3} \left( \epsilon r^2 + L^2 \right) (r - 2G_N M)$$

$$r < 2G_N M \implies V(r) < 0$$

Particle cannot scape the gravitational potential unless is traveling faster than light

For a star one hits the surface before $r = 2G_N M$

A black hole is such that matter has collapse inside its Schwarzschild radius, $r_s = 2G_N M$
Two singularities

1) \( r = 0 \) True singularity \( \Rightarrow \) \( R, R_{\mu\nu}, R_{\mu\nu\sigma\rho} \) diverge

2) \( r = 2G_N M \) Coordinate singularity \( \Rightarrow \) bad frame

\[
\begin{align*}
\text{If} & \quad u = \left( \frac{r}{2G_N M} - 1 \right)^{1/2} e^{r/4G_N M} \cosh \left( \frac{t}{4G_N M} \right) \\
& \quad v = \left( \frac{r}{2G_N M} - 1 \right)^{1/2} e^{r/4G_N M} \sinh \left( \frac{t}{4G_N M} \right)
\end{align*}
\]

\[
\begin{align*}
ds^2 &= \frac{32(G_N M)^3}{r} e^{-r/2G_N M} (-du^2 + dv^2) + r^2 d\Omega_2^2
\end{align*}
\]

Nothing happens at \( r_s = 2G_N M \)
Schwarzschild solution

However, something special still happens at $r_s = 2G \eta M$

Sees (1) moving slower and slower, taking infinite time to reach $r_s = 2G \eta M$
However, something special still happens at $r_s = 2G_N M$

(1) takes a finite time to cross $r_s = 2G_N M$ then $r$ and $t$ swap roles and (1) goes straight into $r=0$

Does (1) moving slower and slower, taking infinite time to reach $r_s = 2G_N M$
However, something special still happens at $r_s = 2G_N M$. 

Schwarzschild solution
Penrose diagram

Schwarzschild solution
Schwarzschild solution

Penrose diagram (completion of spacetime)
Kerr BH

Add rotation

\[ ds^2 = - \left(1 - \frac{rr_s}{\rho}\right) dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 \]
\[ + \left(r^2 + \alpha^2 + \frac{r_s r \alpha}{\rho^2} \sin^2(\theta)\right) \sin^2(\theta) d\phi^2 + \frac{2 r_s r \alpha \sin^2(\theta)}{\rho^2} dt d\phi \]

\[ \alpha = \frac{J}{M} \]
\[ \Delta = r^2 - rr_s + \alpha^2 \]
\[ \rho^2 = r^2 + \alpha^2 \cos^2(\theta) \]
Kerr BH

Momentum bound

\[ J^2 \leq M^4 \]

Extremal case

\[ J \sim M^2 \]

(BPS, stable, in nature?)
**Reissner–Nordström BH**

**Electric charge**

\[ r_Q^2 = \frac{Q^2 G_N}{4\pi \varepsilon_0} \]

\[ ds^2 = -\left(1 - \frac{r_s}{r} - \frac{r_Q^2}{r^2}\right) dt^2 + \left(1 - \frac{r_s}{r} - \frac{r_Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_2^2 \]

**Extremal case when** \(2r_Q \sim r_s\)

*Charged+Rotating* \(\rightarrow\) *Kerr-Newman solution*
Kerr, KN or RN
Kerr, KN or RN

New Parallel Universe

New Universe

White Hole

Parallel Horizon

Parallel Inner Horizon

Parallel Horizon

Parallel Inner Horizon

Parallel Horizon

Parallel Inner Horizon

Parallel Horizon

Parallel Inner Horizon

Black Hole

Universe

Parallel Universe
Black holes have no hair

If matter collapses into a black hole, then all information would be lost, since the later is described by its Mass, Charge and Angular Momentum only.

Moreover, due to quantum processes, it emits isotropic (Hawking) radiation in a rate proportional to its mass ($M$). Therefore, it looses energy and evaporates in a time of the order $t \sim M^3$.

The associated radiation temperature is related to its surface gravity (acceleration of observer on its horizon).
## BH thermodynamics

<table>
<thead>
<tr>
<th></th>
<th><strong>Thermodynamics</strong></th>
<th><strong>Black Holes</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>0th – law</td>
<td>Constant $T$</td>
<td>Constant surface gravity $\kappa$ (at horizon)</td>
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<tr>
<td></td>
<td>in thermal eq.</td>
<td></td>
</tr>
<tr>
<td>1st – law</td>
<td>$dU = \delta Q + \delta W$</td>
<td>$dM = \frac{\kappa}{8\pi} dA + \Omega dJ + \Phi dQ$</td>
</tr>
<tr>
<td>2nd – law</td>
<td>$\frac{dS}{dt} \geq 0$</td>
<td>$\frac{dA}{dt} \geq 0$</td>
</tr>
<tr>
<td>3rd – law</td>
<td>No system with $S=0$</td>
<td>No BH with $\kappa = 0$</td>
</tr>
<tr>
<td></td>
<td>in finite series of steps</td>
<td></td>
</tr>
</tbody>
</table>
Implication

Temperature \quad \text{Entropy} \quad \text{Surface gravity} \quad \text{Area of horizon}

\[
S = \frac{1}{4} A
\]
Cosmology

Observations

Hot Big Bang Theory

Early Universe ideas
Cosmology

Until the XX century, the Universe was though to be finite and as big as the Milky Way

*(problem: how to measure distances)*

Distance ladder

Cepheids

Real luminosity depends on period
Apparent luminosity provides distance
Cosmology

Now we know:

0) Dark
1) Bigger
2) Expanding

Velocity = H (distance)
Cosmology

Implies a denser/hotter past

Cosmic Microwave Background (CMB)

\[ T = 2.7 \, K \]
Cosmology

Small perturbations of 1 in 100000

Simple UNIVERSE

Gaussian
Scale invariant (tilt)

COBE
WMAP
Planck
Cosmology

Power spectrum (two point correlation function)
Cosmology

Power spectrum (two point correlation function)

\[ G_{\mu\nu} = 8\pi G_N T_{\mu\nu} \]
Large Scale Structure

On sufficiently large scales (>200Mpc) the Universe is isotropic and homogeneous.

Filaments
Voids
But, what else can we say about its energy/matter content?
Emitted light does not say it all...
Dark Matter

Up to 80% of matter in the Universe today is dark!
Dark Matter

Gravitational lensing
¡We do not understand what it is!

Only acts gravitationally and does not emit light

Best candidate: a weakly interacting particle that we have not seen yet

Or have we?
Some experiments (Dama, CoGeNT, CDMS) have signals which are hard to explain with known physics, but others (Xenon) have not seen anything in the same regions.
Dark Matter

Or, could it be a modification of gravity?
Dark Energy

There are other objects (called supernovae IA) which also belong to the distance ladder. The supernovae are big star explosions.

Remanente de SN 1572

SN 1987A
Dark Energy

Hubble diagram:
Supernovae tell us the Universe presents accelerated expansion
Dark Energy

We understand even less what it is!

Only acts gravitationally, but in a repulsive way

Best candidate: Cosmological Constant
Dark Energy

But...

Quantum vacuum gravitates as a cosmological constant (zero point energy)

Theory = 1000000000000000000000000000000000000000000000000000000000000000

Observation!!
Cosmic Pie

- 68% Energía Obs.
- 27% Materia Obs.
- 5% Materia Visible

Legend:
- Blue: Materia Visible
- Red: Materia Obs.
- Yellow: Energía Obs.
Theory

Cosmological Principle

Assume an isotropic and homogeneous metric

\[ ds^2 = -dt^2 + a^2(t)(dr^2 + f(r)d\Omega_2^2) \]

Scale factor

\[ f(r) = \begin{cases} 
\sinh^2(r) & \text{Open} \\
r^2 & \text{Flat} \\
\sin^2(r) & \text{Closed} 
\end{cases} \]
Friedmann equations

Einstein equations with a perfect fluid reduce to

\[ G_{00} \Rightarrow H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \frac{8 \pi G_N}{3} \rho - \frac{k}{a^2} \]

\[ G_{ij} \Rightarrow \dot{H} + H^2 = -\frac{4 \pi G_N}{3} (\rho + 3p) \]

\[ G_{i0} = 0 \]
Critical Mass

\[ k = 0 \quad \Rightarrow \quad \rho_c(t) = \frac{3H^2}{8\pi G_N} \]

Today's critical mass is

\[ \rho_c^0 = \frac{3H_0^2}{8\pi G_N} \sim 10^{-29} \frac{g}{cm^3} \]

Where \( H_0 = 71 Km/s/Mpc \) according to WMAP
The Theory

Define in terms of critical mass today (and including a cosmological constant)

\[ \Omega_\rho = \frac{\rho}{\rho_c^0} \quad \Omega_k = -\frac{k}{a^2 H_0^2} \quad \Omega_\Lambda = \frac{\Lambda}{3 H_0^2} \]

Friedmann eqn

\[ H^2 = \frac{8\pi G_N}{3} \rho - \frac{k}{a^2} + \frac{\Lambda}{3} \]

reduce today to

\[ \frac{H^2(t_0)}{H_0^2} = 1 = \Omega_\rho + \Omega_k + \Omega_\Lambda \]
The general solution for a flat spatial slicing \((k=0)\) and a perfect fluid with equation of state

\[ \rho = \omega \rho \]

where the equations reduce to

\[ H^2 = \frac{8\pi G_N}{3} \rho \]
\[ \dot{H} + H^2 = -\frac{4\pi G_N}{3} (3 + \omega) \rho \]

is

\[ a(t) = a_0 t^{\frac{2}{3(\omega+1)}} \]
Theory

1) Dust \((p=0)\)

\[ a(t) = \left( \frac{3H_0}{2} \right)^{2/3} t^{2/3} \]

Age of the Universe

\[ a(t_0) = 1 = \left( \frac{3H_0}{2} \right)^{2/3} t_0^{2/3} \quad \Rightarrow \quad t_0 = \frac{2}{3H_0} \]

2) Radiation \((p=1/3)\)

\[ a(t) = \left( 2H_0 \right)^{1/2} t^{1/2} \]
Solutions with curvature ($k$)

\[ H^2 = \frac{8\pi G_N}{3} \rho - \frac{k}{a^2} \]
Theory

Cosmological constant domination

\[ H^2 = \frac{\dot{a}^2}{a^2} = \frac{\Lambda}{3} \]

\[ a(t) = A_0 e^{\frac{\Lambda t}{3}} \]
Scale factor evolution

\[ p = w \rho \]

\[ a(t) = a_0 t^{\frac{1}{3(w+1)}} \]
Hot Big Bang

Scale factor evolution

\[ p = w \rho \]

\[ a(t) = a_0 t^{\frac{1}{3(w+1)}} \]

Model fits ALL observations to great accuracy

- Dark energy
- Radiation
- \( e^{H_0 t} \)
Other observations support this model (age of stars, light element abundances, etc).
Hot Big Bang

Problems

1) Magnetic monopoles
   With many phase transitions why there are any topological relics?

2) Horizon problem
   Why disconnected region of space have same CMB temperature

3) Size
   $l_{lp}^3 \xrightarrow{\text{Expansion}} 13.7 \text{ Gyrs} \neq 10^{28} \text{ m}^3$

4) Flatness
Hot Big Bang

Problems

1) Magnetic monopoles

Why there are any topological relics?

2) Horizon problem

Why disconnected regions of space have the same CMB temperature?

3) Size

4) Flatness

5) The initial (big bang) singularity

\[ t \to 0 \implies a(t) \to 0 \]
\[ T, \rho, R \to \infty \]

13.7 Gyr

1 mm³

10^{28} m³

\( l_{lp}^3 \)
Inflation

Phase of exponential acceleration

Simplest realisation: canonical scalar field \textit{slowly} rolling down a potential

\[
S = \int d^4 x \sqrt{-g} \left( \frac{m_{pl}^2}{2} R - \frac{1}{2} (\partial \phi)^2 - V(\phi) \right)
\]

\[
\rho = \frac{1}{2} \dot{\phi}^2 + V(\phi), \quad p = \frac{1}{2} \dot{\phi}^2 - V(\phi).
\]

\[
w = \frac{\dot{\phi}^2 + V(\phi)}{\dot{\phi}^2 - V(\phi)}
\]
Friedmann equations read
\[ H^2 = \frac{1}{3m_{pl}^2} \left[ \frac{1}{2} \dot{\phi}^2 + V(\phi) \right], \quad \ddot{\phi} + 3H \dot{\phi} = -V'(\phi). \]

If slow-roll is assumed (potential energy dominates over kinetic), then
\[ H^2 \approx \frac{V(\phi)}{3m_{pl}^2}, \quad 3H \dot{\phi} \approx -V'(\phi) \]

which imply the following solution
\[ a(t) = a(t_0) \exp \left[ \int_{t_0}^{t} H dt \right] \sim a(t_0) e^{Ht} \]
Inflation

Solves the Problems

1) Magnetic monopoles
   The exponential expansion dilutes them

2) Horizon problem
   A small patch was enlarged beyond the Hubble horizon
   (ie got causally disconnected because of the expansion)

3) Size

4) Flatness
   Exponential expansion decreases curvature, if any

5) Singularity problem remains
In order to solve problems inflation should last

\[ N = 50 - 60 \text{ e-foldings} \quad N(t) = \ln[a(t_{\text{end}})/a(t_{\text{start}})] \]

**Slow-roll parameters**

\[ \epsilon_s \equiv m_{\text{pl}}^2 \frac{\phi V}{2V^2}, \quad \eta_s \equiv m_{\text{pl}}^2 \frac{\phi V}{V} \]

Should be roughly <0.01 to achieve inflation.

Measure of when the approximation breaks down (inflation ends). After that one needs a mechanism of reheating.
Inflation

“Quantum” fluctuations

Scalar perturbations

\[ ds^2 = a^2(\tau) \left[-(1 + 2\Gamma)d\tau^2 - 2\nabla_i \Omega dx^i d\tau \right. \]
\[ \left. + [(1 - 2\Psi)\delta_{ij} - 2\nabla_i \nabla_j \xi] dx^i dx^j \right] \]

\[ \phi = \phi_0(\tau) + \delta \phi(\tau, x^i) \]

Changing variables

\[ z = \frac{a \phi_0}{\mathcal{H}}, \quad u = a \left[ \delta \phi + \frac{\phi_0'}{\mathcal{H}} \Psi \right] \]

\[ u'' - \nabla_i \nabla^i u - \frac{z''}{z} u = 0 \]
In slow-roll

\[ \frac{z''}{z} \sim \mathcal{H}^2 \]

It is a harmonic oscillator! Easy to solve providing consistent initial conditions

The curvature perturbations (which eventually get imprinted in the CMB) are

\[ \mathcal{R} = -\frac{u}{z} = \Psi + \frac{\mathcal{H}}{\phi_0'} \delta \phi \]

We want to calculate the two-point correlation function

\[ \langle \mathcal{R}_k \mathcal{R}_{k'} \rangle = \frac{2\pi^2}{k^3} \mathcal{P}_\mathcal{R}(k) \delta(k - k') \]
Inflation

Where $k$ is the wavenumber of the perturbation.

After some algebra, one obtains

$$P(k, t) = \left( \frac{H^2}{\dot{\phi}_0} \right)^2$$

which is constant to leading order in slow-roll. Therefore, the power spectrum is scale-invariant, and corrections to that are given by the slow-roll parameters. These results can be contrasted with the CMB, fitting very well the data.
Inflation

As a bonus, it explains CMB anisotropies

But introduces new (or keeps some) problems:

It does not explain the amplitude of the CMB anisotropies

What is the “inflaton” (inflation's scalar field)?

Why did we start at the top of the potential?

Initial conditions (singularity)?
However, Planck's results put some pressure
Cyclic Model

Alternative model where a Pre-Big bang phase creates the CMB patterns.

Two flavours: singular & non-singular
Cyclic Model
Cyclic model

Creates perturbations in a very similar way to inflation

But... again some pressure by Planck
End of the course