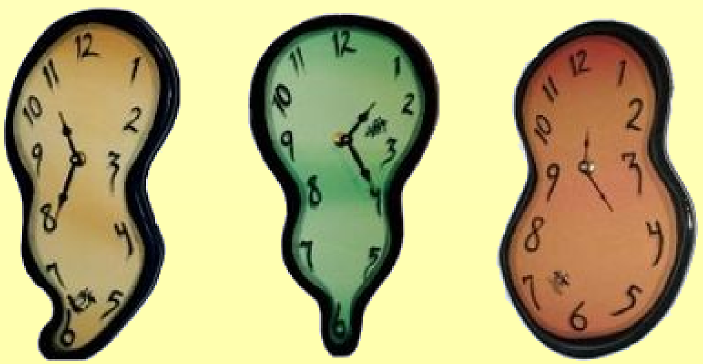
En búsqueda del mundo cuántico de la gravedad



Escuela de Verano 2015

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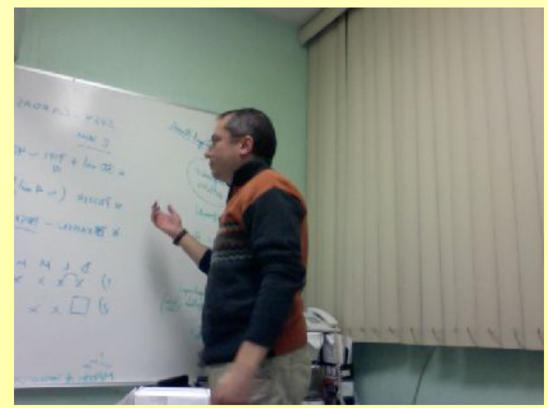


Hoy y Viernes – Mayor información

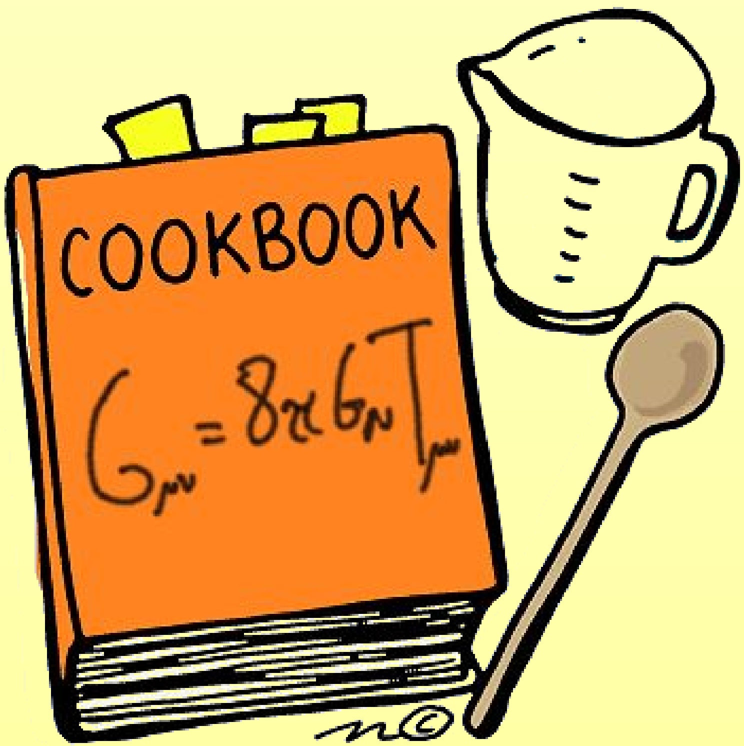
Quantum gravity. Part I

1. General Relativity

2. Quantum field Theory

3. Where is the UV problem?

GR as a Cooking Book



- Ingredients
  - Physics & Maths
- Cookware
  - Equations
- Recipes
  - Solutions

Ingredients

- Why GR?

Newtonian gravity ➡ Particle moves in straight line until hit by a force

Vs

Einstein's gravity ➡ Gravity is not a force but a result of space-time geometry

Ingredients

- Why GR?

Newtonian gravity ➡ Particle moves in straight line until hit by a force

Vs

Einstein's gravity (GR) ➡ Gravity is not a force but a result of space-time geometry

Keywords

Ingredients

Need to understand

a) Space and time together

b) Curved space

Ingredients

Need to understand

a) Space and time together

b) Curved space

Tensor Notation

Ingredients: Special Relativity

- Speed of light is constant in any frame (c=1)
- Space and time are entangled

Coordinates of space-time

$$x^\mu$$

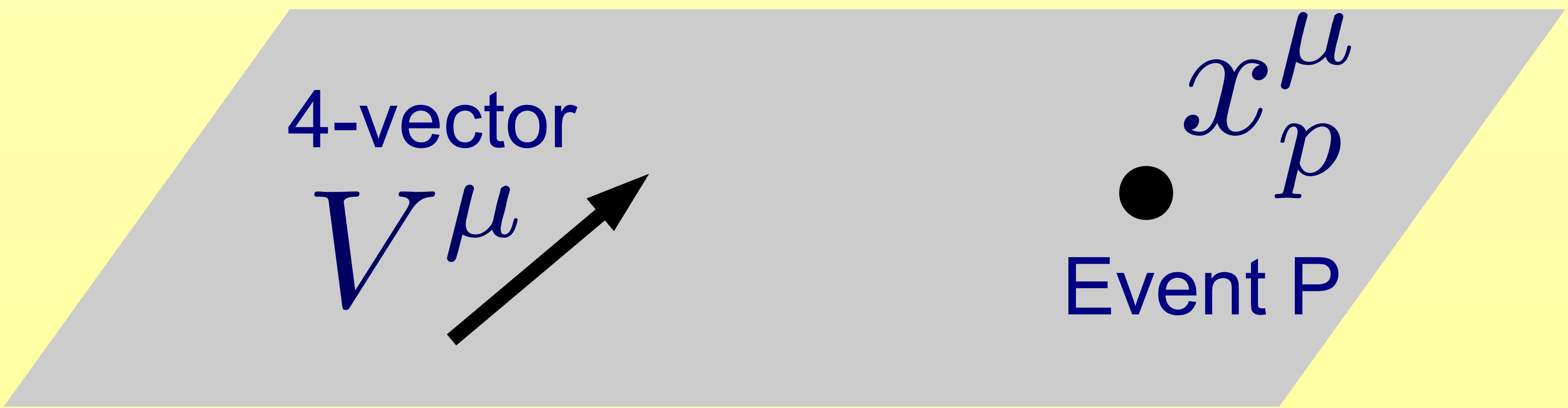
$$x^0 \equiv ct = t$$

$$\left.\begin{aligned} x^1 &\equiv x \\ x^2 &\equiv y \\ x^3 &\equiv z \end{aligned}\right\} x^i$$

Ingredients: Special Relativity

Special Relativity (SR) lives in

Minkowski space-time



Ingredients: Special Relativity

Minkowski metric

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Distance (dot product)

Summation convention

$$A \cdot B = \eta_{\mu\nu} A^\mu B^\nu = -A^0 B^0 + A^1 B^1 + A^2 B^2 + A^3 B^3$$

Infinitesimal distance (line element)

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -dt^2 + dx^2 + dy^2 + dz^2$$

Same information as in the metric

Ingredients: Special Relativity

Lorentzian signature **(- , + , + , +)**

[sometimes (+,-,-,-)]

C.f. Euclidean 4d space $\delta_{\mu\nu}$ instead of $\eta_{\mu\nu}$

$$ds_E^2 = \delta_{\mu\nu} dx^\mu dx^\nu = dw^2 + dx^2 + dy^2 + dz^2$$

Euclidean signature **(+ , + , + , +)**

Ingredients: Special Relativity

Lorentz group

– Transformations that leave ***ds*** unchanged

e.g. x-direction boost

$$t \to \gamma(t - v\,x)$$
$$x \to \gamma(x - v\,t)$$

$$\gamma \equiv (1 - v^2)^{-1/2}$$

Exercise

$$ds^2 \to ds^2$$

Ingredients: Special Relativity

For a particle moving only in time (***x^i=const.***) the elapsed time is

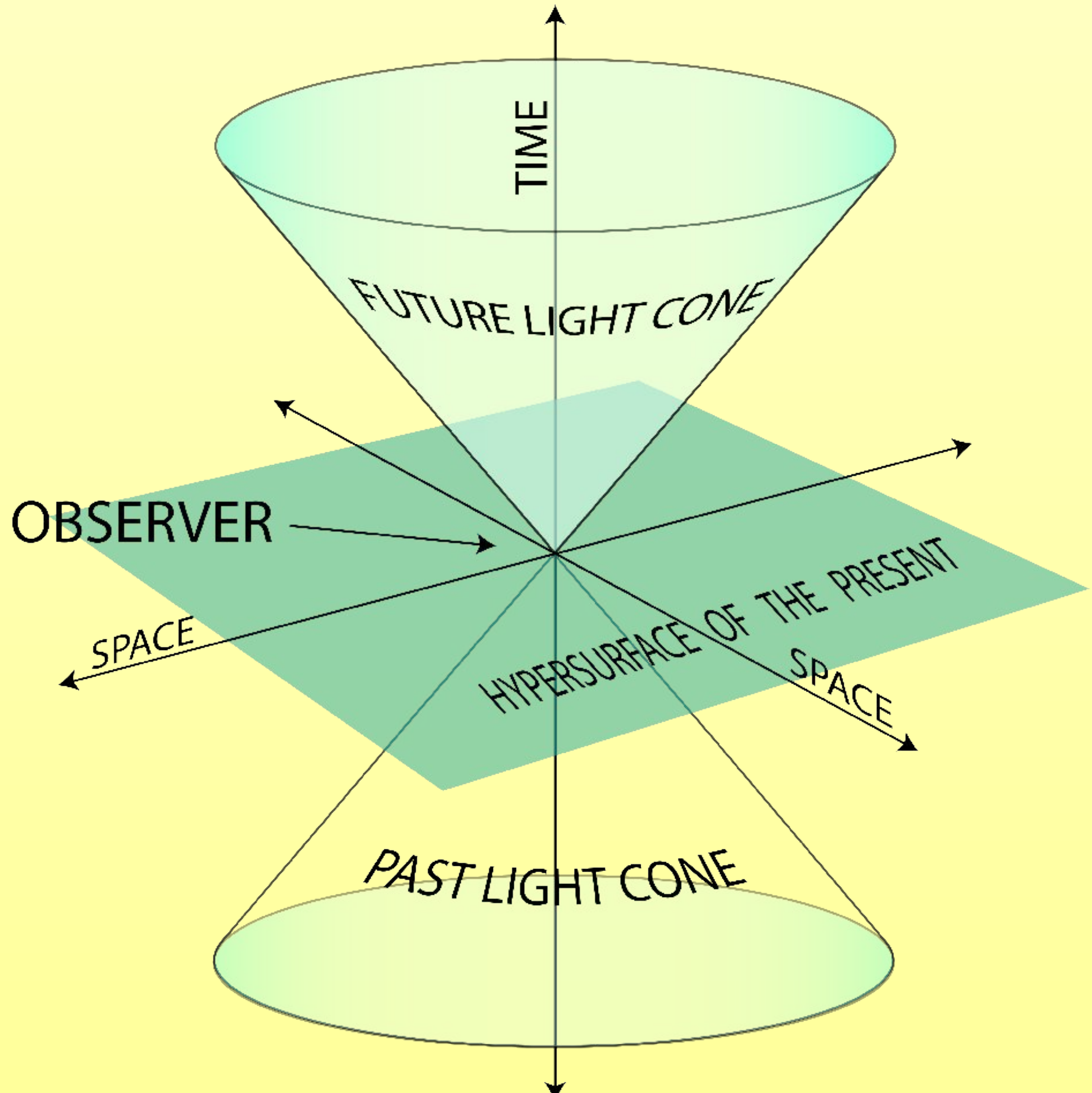
$$ds^2 = -dt^2 < 0$$

Define proper time as

$$\tau = \int \sqrt{-ds^2}$$

Ingredients: Special Relativity

Spacetime diagram



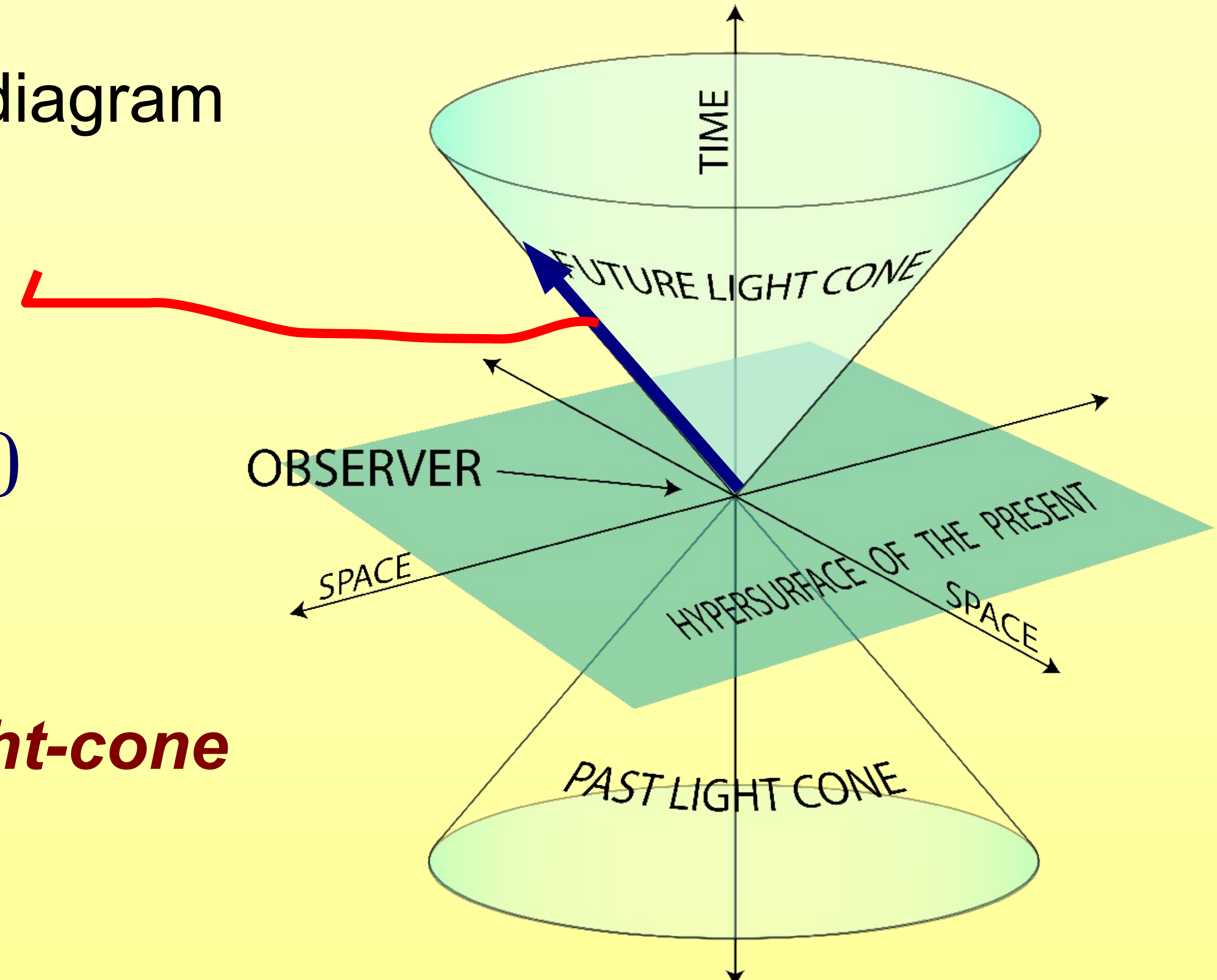
Ingredients: Special Relativity

Spacetime diagram

Null vector

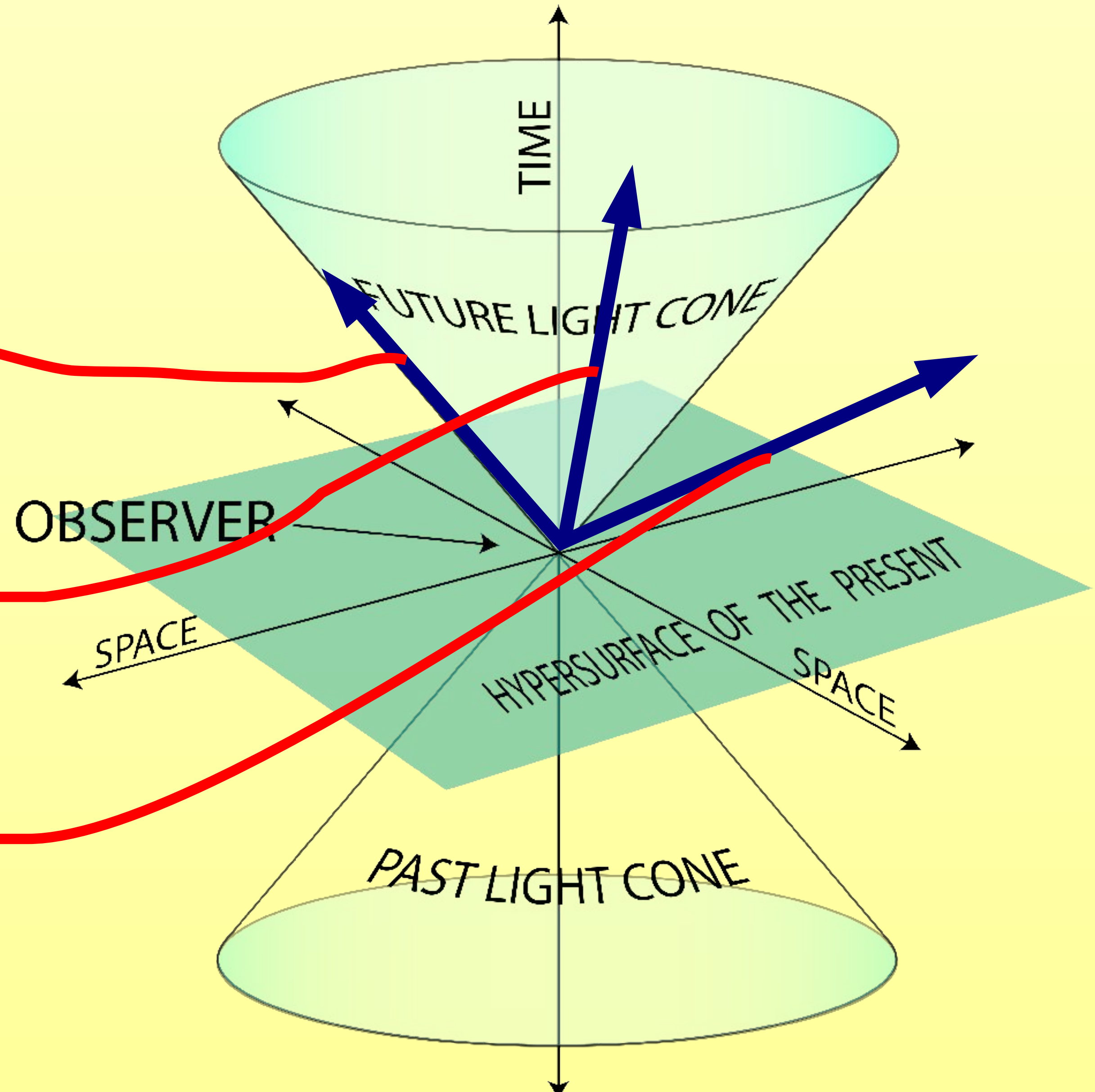
$$ds^2 = 0$$

Defines light-cone



Ingredients: Special Relativity

Spacetime diagram



Null vector

$$ds^2 = 0$$

Timelike vector

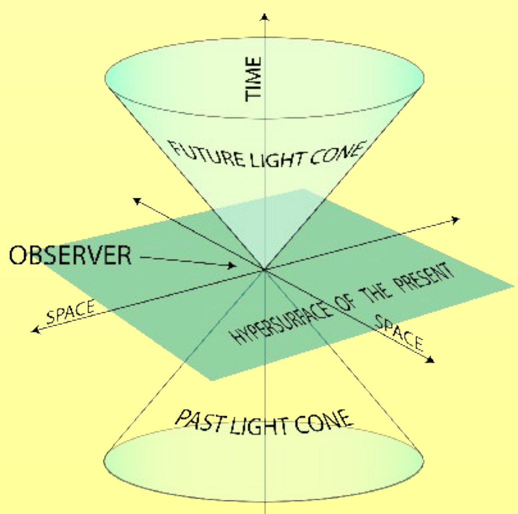
$$ds^2 < 0$$

Spacelike vector

$$ds^2 > 0$$

Ingredients: Special Relativity

$$x^\mu(\lambda)$$



Trajectory specified by

$$x^\mu(\lambda)$$

is timelike/spacelike/null

if tangent vector $\dfrac{dx^\mu}{d\lambda}$

is timelike/spacelike/null

Ingredients: Special Relativity

- Null trajectories represent light or massless particles
- Spacelike trajectories are causally disconnected points (or particles with $v>1$)
- Timelike trajectories describe massive particles (observers)

More convenient to use proper time

$$\tau = \int \sqrt{-ds^2} = \int \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda$$

Ingredients: Special Relativity

Massive particles

4-velocity

$$\mathcal{U}^\mu = \frac{dx^\mu}{d\tau}$$

$$\left(\mathcal{U}^\mu \mathcal{U}_\mu = \eta_{\mu\nu}\mathcal{U}^\mu \mathcal{U}^\nu = -1\right)$$

4-momentum

$$P^\mu = m\,\mathcal{U}^\mu$$

m **rest frame mass**

Ingredients: Special Relativity

Massive particles

4-velocity

$$\mathcal{U}^\mu = \frac{dx^\mu}{d\tau}$$

4-momentum

$$P^\mu = m\,\mathcal{U}^\mu$$

Energy is P^0, which in the rest frame, is

$$P^0 = m \qquad \left(\text{c.f. } E = mc^2\right)$$

Ingredients: Special Relativity

Massive particles

One can boost the frame with a Lorentz transf., say in the x-dir.

$$\gamma \equiv (1 - v^2)^{-1/2}$$

$$P^\mu = (\gamma\, m, v\, \gamma\, m, 0, 0)$$

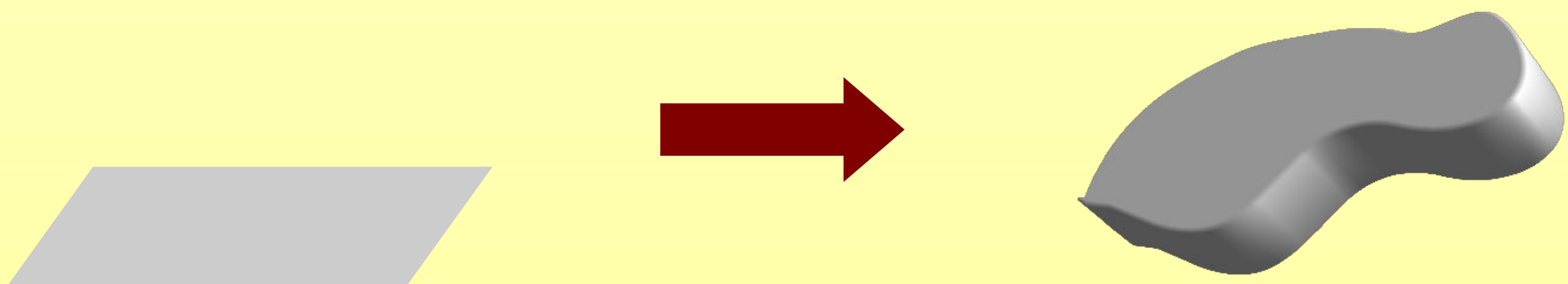
Which for small ***v*** reduces to

$$P^0 = m + \frac{1}{2} m\, v^2$$ **Rest mass + kinetic energy**

$$P^1 = m\, v$$ **Newtonian momentum**

Ingredients: Curved Space

Minkoswki | Manifold



Cartesian coords. | Cannot use Cartesian coords. globally

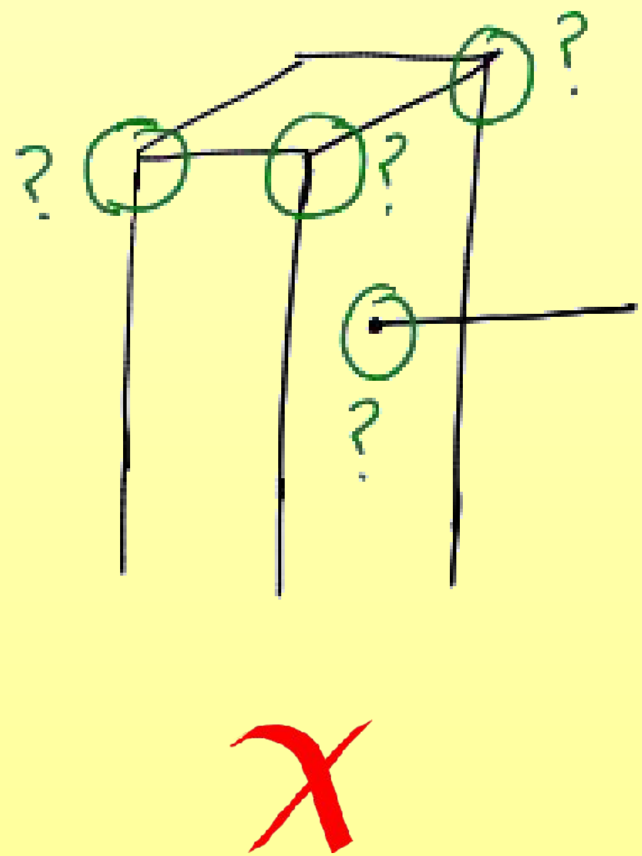
Ingredients: Curved Space

Manifold

Def. A topological space $\mathcal{M}$ is an n-dimensional manifold if there is a collection (atlas, $\{\phi_i\}$) of maps (charts, ϕ_i) , such that each map $\phi_i : \mathcal{M} \to \Re^n$ is continuous, bijective and invertible

Ingredients: Curved Space

In simple words



Ingredients: Curved Space

Equivalence Principle.

A choice of chart (coordinates) is arbitrary and the physics should *not depend* on this!

A convenient way to describe chart-independent equations is to use ***tensors***, which can be thought of generalisations of *vectors*, with possibly more indices

Ingredients: Tensors

Under a coordinate transformation $x^\mu \to x'^\mu$ a vector A^μ changes in the following way

$$A'^\mu = \frac{\partial x^\mu}{\partial x'^\nu} A^\nu$$

(summation convention)

Ingredients: Tensors

Under a coordinate transformation $x^\mu \rightarrow x'^\mu$ a tensor $T_{\mu_1,\ldots,\mu_n}{}^{\nu_1,\ldots,\nu_m}$ changes in the following way

(n,m) - Tensor

$$T'_{\mu_1,\ldots,\mu_n}{}^{\nu_1,\ldots,\nu_m} = \frac{\partial x'^{\bar{\mu}_1}}{\partial x^{\mu_1}} \cdots \frac{\partial x'^{\bar{\mu}_n}}{\partial x^{\mu_n}} \frac{\partial x^{\nu_1}}{\partial x'^{\bar{\nu}_1}} \cdots \frac{\partial x^{\nu_m}}{\partial x'^{\bar{\nu}_m}} T_{\bar{\mu}_1,\ldots,\bar{\mu}_n}{}^{\bar{\nu}_1,\ldots,\bar{\nu}_m}$$

(summation convention)

Ingredients: Tensors

Observations

Definitions

(0,0) – tensor ➡ scalar (function)

(0,1) – tensor ➡ vector

(1,0) – tensor ➡ co-vector

Ingredients: Tensors

Observations

Maths

Trace over indices $S^{\mu} = T^{\lambda\mu}{}_{\lambda}$

Symmetric $S_{\dots\alpha\beta\dots} = +S_{\dots\beta\alpha\dots}$

Antisymmetric $A_{\dots\alpha\beta\dots} = -A_{\dots\beta\alpha\dots}$

Ingredients: Tensors

Observations

Physics

An equation that holds in ***one*** coordinate system holds in ***all*** coordinate systems

Equivalence Principle!

Ingredients: Tensors

In GR the most important tensor is the metric

$$g_{\mu\nu}$$

a generalisation of Minkowski's metric $\eta_{\mu\nu}$ which encodes the geometrical information of spacetime

Ingredients: Tensors

At a given point, there is always a coordinate system such that

$$g_{\mu\nu}(x_p^\mu) = \eta_{\mu\nu}$$

i.e. the spacetime looks locally flat

and also first derivatives of $g_{\mu\nu}$ vanish!

Second derivatives cannot be made to vanish,

a manifestation of ***curvature***

Ingredients: Tensors

Dot product $$A \cdot B \equiv g_{\mu\nu} A^\mu B^\nu$$

Suggests the concept of raising/lowering indices, namely

$$A_\mu = g_{\mu\lambda} A^\lambda \qquad B^\mu = g^{\mu\lambda} B_\lambda$$

where $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$

so $g_{\mu\lambda} g^{\lambda\nu} = \delta_\mu^\nu$

Ingredients: Tensors

Important objects in physics which are not tensors

1) determinants

$$\text{e.g.} \qquad g \equiv \det g_{\mu\nu}$$

Under $x^\mu \to x'^\mu$ transforms as

$$g \to \left[\det\left(\frac{\partial x^\mu}{\partial x'^\nu}\right)\right]^{-2} g$$

Ingredients: Tensors

Important objects in physics which are not tensors

2) volume factor $$d^4x = dx^0\,dx^1\,dx^2\,dx^3$$

Transforms as $$d^4x \to \det\left(\frac{\partial x^\mu}{\partial x'^\nu}\right) d^4x$$

1) and 2) are call ***tensor densities*** because transform as some powers of the Jacobian

Ingredients: Tensors

Important objects in physics which are not tensors

Notice

$$\int f(x^\mu)\sqrt{-g}d^4x$$

Is invariant under a change of coordinate, since ***f(x)*** is a scalar

Ingredients: Tensors

Important objects in physics which are not tensors

Notice

$$\int f(x^\mu)\sqrt{-g}d^4x$$

Is invariant under a change of coordinate, since ***f(x)*** is a scalar

Therefore the right way of writing integrals is using

$$\int d^4x \to \int \sqrt{-g}d^4x$$

Ingredients: Tensors

Important objects in physics which are not tensors

3) Partial derivatives

On scalar are OK $$\partial_\mu \phi \to \partial_{\bar{\mu}} \phi' = \frac{\partial x^\mu}{\partial x'^{\bar{\mu}}} \partial_\mu \phi$$

...but on vectors,

$$\partial_\mu V^\nu \to \partial_{\bar{\mu}} V'^{\bar{\nu}} = \frac{\partial x^\mu}{\partial x'^{\bar{\mu}}} \partial_\mu \left[\frac{\partial x'^{\bar{\nu}}}{\partial x^\nu} V^\nu \right]$$

$$= \frac{\partial x^\mu}{\partial x'^{\bar{\mu}}} \frac{\partial x'^{\bar{\nu}}}{\partial x^\nu} \partial_\mu V^\nu \; + \; \frac{\partial x^\mu}{\partial x'^{\bar{\mu}}} \frac{\partial^2 x'^{\bar{\nu}}}{\partial x^\mu \partial x^\nu} V^\nu$$

Ingredients: Tensors

Important objects in physics which are not tensors

$$T'{}_{\mu_1,\ldots,\mu_n}{}^{\nu_1,\ldots,\nu_m}$$

$$= \frac{\partial x'^{\bar\mu_1}}{\partial x^{\mu_1}} \cdots \frac{\partial x'^{\bar\mu_n}}{\partial x^{\mu_n}} \frac{\partial x^{\nu_1}}{\partial x'^{\bar\nu_1}} \cdots \frac{\partial x^{\nu_m}}{\partial x'^{\bar\nu_m}} T_{\bar\mu_1,\ldots,\bar\mu_n}{}^{\bar\nu_1,\ldots,\bar\nu_m}$$

$$\partial_\mu V^\nu \to \partial_{\bar\mu} V'^{\bar\nu} = \frac{\partial x^\mu}{\partial x'^{\bar\mu}} \partial_\mu \left[\frac{\partial x'^{\bar\nu}}{\partial x^\nu} V^\nu \right]$$

$$= \underline{\frac{\partial x^\mu}{\partial x'^{\bar\mu}} \frac{\partial x'^{\bar\nu}}{\partial x^\nu} \partial_\mu V^\nu} + \underline{\frac{\partial x^\mu}{\partial x'^{\bar\mu}} \frac{\partial^2 x'^{\bar\nu}}{\partial x^\mu \partial x^\nu} V^\nu}$$

✓ ✗

Ingredients: Tensors

Define the ***Covariant Derivative***

$$\nabla_\mu V^\nu \equiv \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda$$

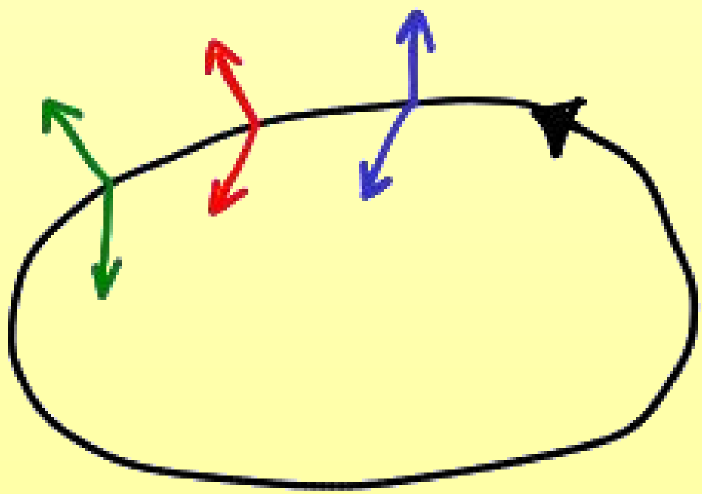
Connection → *transforms to cancel this*

$$\partial_{\bar{\mu}} V'^{\bar{\nu}} = \frac{\partial x^\mu}{\partial x'^{\bar{\mu}}} \frac{\partial x'^{\bar{\nu}}}{\partial x^\nu} \partial_\mu V^\nu + \frac{\partial x^\mu}{\partial x'^{\bar{\mu}}} \frac{\partial^2 x'^{\bar{\nu}}}{\partial x^\mu \partial x^\nu} V^\nu$$

✓ ✗

Ingredients: Tensors

$\nabla_\mu V^\nu$ transforms a tensor and defines *parallel transport*



Ingredients: Tensors

$\nabla_\mu V^\nu$ transforms a tensor and

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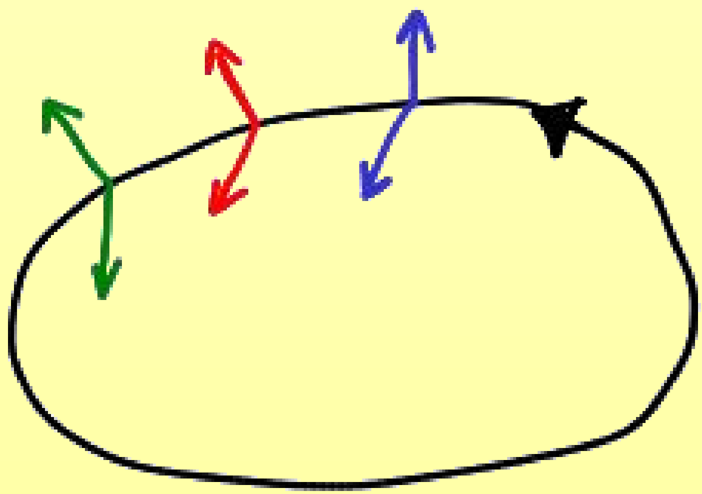


Similarly for lower indices

$$\nabla_\mu w_\nu \equiv \partial_\mu w_\nu - \Gamma^\lambda_{\mu\nu} w_\lambda$$

Ingredients: Tensors

Christoffel Symbols

There is a particular choice of connection that

$$\nabla_\mu (g_{\nu\sigma}) = 0$$

In components, it is

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right)$$

Ingredients: Curvature

Information about the *curvature* is contained in the *metric tensor*, and it is the ***Riemann tensor*** which explicitly accounts for it

$$R^{\sigma}{}_{\mu\alpha\beta} = \partial_{\alpha}\left(\Gamma^{\sigma}{}_{\mu\beta}\right) - \partial_{\beta}\left(\Gamma^{\sigma}{}_{\mu\alpha}\right) + \Gamma^{\sigma}{}_{\alpha\lambda}\Gamma^{\lambda}{}_{\mu\beta} - \Gamma^{\sigma}{}_{\beta\lambda}\Gamma^{\lambda}{}_{\mu\alpha}$$

(note: it has second derivatives of the metric)

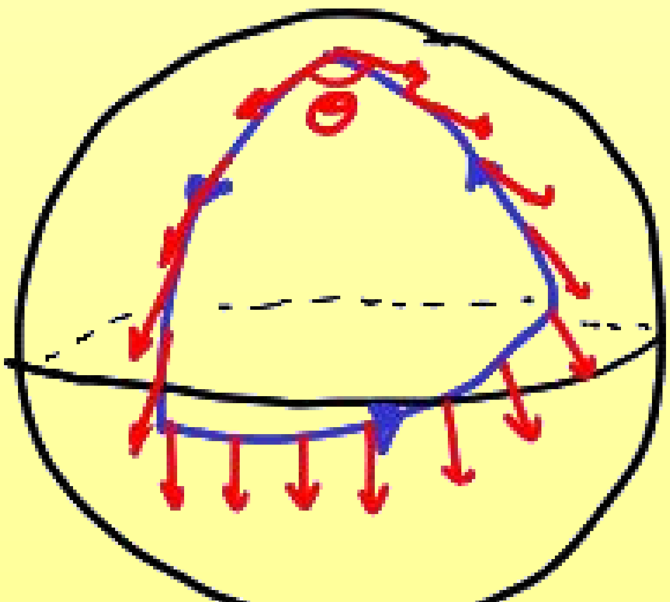
Ingredients: Curvature

Geometrical interpretation

In Euclidean space if a vector is parallel transported around a closed loop, it returns unchanged.

In curved space, this is not necessarily true.

The Riemann tensor measures the difference!



Ingredients: Curvature

Properties of $R^{\sigma}_{\mu\nu\lambda}$

1) $R_{\mu\nu\lambda\sigma}$

Antisymmetric

Symmetric

2) $\nabla_{\rho} R_{\mu\nu\lambda\sigma} + \nabla_{\nu} R_{\rho\mu\lambda\sigma} + \nabla_{\mu} R_{\nu\rho\lambda\sigma} = 0$

(Bianchi identity)

3) $R^{\sigma}{}_{\mu\nu\lambda} + R^{\sigma}{}_{\nu\lambda\mu} + R^{\sigma}{}_{\lambda\mu\nu} = 0$

Ingredients: Curvature

Other tensors derived from $R^{\sigma}_{\mu\nu\lambda}$

1) Ricci tensor

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$$

It satisfies $R_{\mu\nu} = R_{\nu\mu}$

2) Ricci scalar

$$R = R^{\mu}{}_{\mu} \equiv g^{\mu\nu} R_{\mu\nu}$$

Ingredients: Curvature

Other tensors derived from $R^{\sigma}_{\mu\nu\lambda}$

3) Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

It obeys

$$\nabla_\mu G^{\mu}{}_{\nu} = 0$$

Cookware: Einstein Equations

Einstein (1915)

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu}$$

Cookware: Einstein Equations

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

Energy Momentum tensor

Non-linear equations of the metric components!

Cookware: Einstein Equations

Dimension-full parameters

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu}$$

Newton's Constant $6.67 \times 10^{-11} \mathrm{N} \frac{\mathrm{m}^2}{\mathrm{Kg}^2}$

If c=ħ=1 then $M_{pl}^2 = 8\pi G_N \simeq (10^8 GeV)^2$

Cookware: Einstein Equations

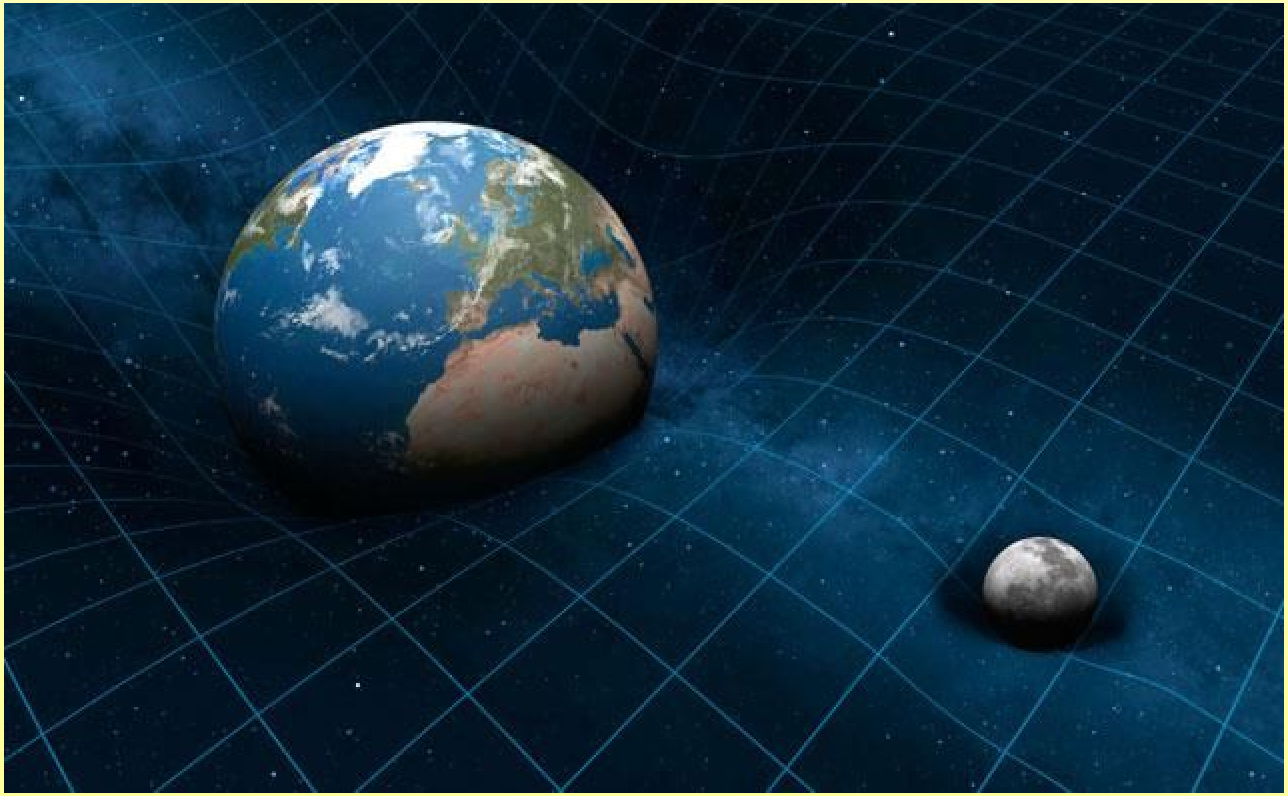
$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu}$$

Geometría del Espacio-tiempo

Materia/Energía

Cookware: Einstein Equations

El espacio-tiempo es como una "manta" invisible deformado por la materia o energía



Cookware: Einstein Equations

Cauchy Problem. Similar for RG

1)	G_{00}	No ***t*** derivatives	Constraints
2)	G_{0i}	1st order ***t*** der.	Constraints
3)	G_{ij}	2nd order ***t*** der.	Evolution eqns.

Find $g_{\mu\nu},\ g_{\mu\nu,0} \equiv \partial_0 g_{\mu\nu}$ for $x^0 = 0$, which satisfy the constraints initially, and use the evolution equations to solve for $x^0 > 0$

The equations are linear in second derivatives!

Cookware: Einstein Equations

Relevant components of the metric

$g_{\mu\nu}$ has 10 independent components (remember it is *symmetric*), but there are 6 evolution equations to determine them (?).

Actually, we have an arbitrary choice of coordinates. Therefore, there are only 10-4=6 variables to determine, which can be found using the evolution equations.

Cookware: Einstein Equations

Energy momentum tensor $T_{\mu\nu}$

A popular choice is a perfect fluid

- Fluid with ***no viscosity***, or ***heat flow***, and ***isotropic*** in its rest frame
- Completely specified by energy density and pressure

$$T_{\mu\nu} = (\rho + p) U_\mu U_\nu - p g_{\mu\nu}$$

$$T^\mu_{\ \nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

where U^μ is the unitary 4-velocity

- Bianchi identity implies conservation

$$\nabla_\mu T^\mu_{\ \nu} = 0$$

Cookware: Einstein Equations

Newtonian limit

Recall Poission equation for the Newtonian potential Φ

$$\nabla^2 \Phi = 4\pi G_N \rho$$

2nd order diff. operator over a field

Cookware: Einstein Equations

Newtonian limit

Recall Poission equation for the Newtonian potential Φ

$$\nabla^2\Phi = 4\pi G_N \rho$$

2nd order diff. operator over a field

In a relativistic theory we expect the energy momentum tensor

$$T_{\mu\nu}$$

Cookware: Einstein Equations

Newtonian limit

Recall Poission equation for the Newtonian potential Φ

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2nd order diff. operator over a field

In a relativistic theory we expect the energy momentum tensor

$$G_{\mu\nu} \quad \text{———} \quad T_{\mu\nu}$$

Einstein equations

Can be done formally

Cookware: Hilbert-Einstein action

Einstein equations can be obtained from

$$S = \int \sqrt{-g} d^4x \left(\frac{1}{16\pi G_N} R + \mathcal{L}_m \right)$$

$$\frac{\delta S}{\delta g_{\mu\nu}} = 0$$ ➡ *Einstein equations*

$$\frac{1}{8\pi G_N}(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g_{\mu\nu}} \equiv T_{\mu\nu}$$

Cookware: Hilbert-Einstein action

Can include a Cosmological Constant

$$S = \int \sqrt{-g} d^4x \left(\frac{1}{16\pi G_N}(R - 2\Lambda) + \mathcal{L}_m \right)$$

$$\frac{\delta S}{\delta g_{\mu\nu}} = 0 \quad \Rightarrow \quad \boxed{G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}}$$

Cookware: Alternative approaches

Hamiltonian (ADM or 3+1 formalism)

Consider the space-time splitting

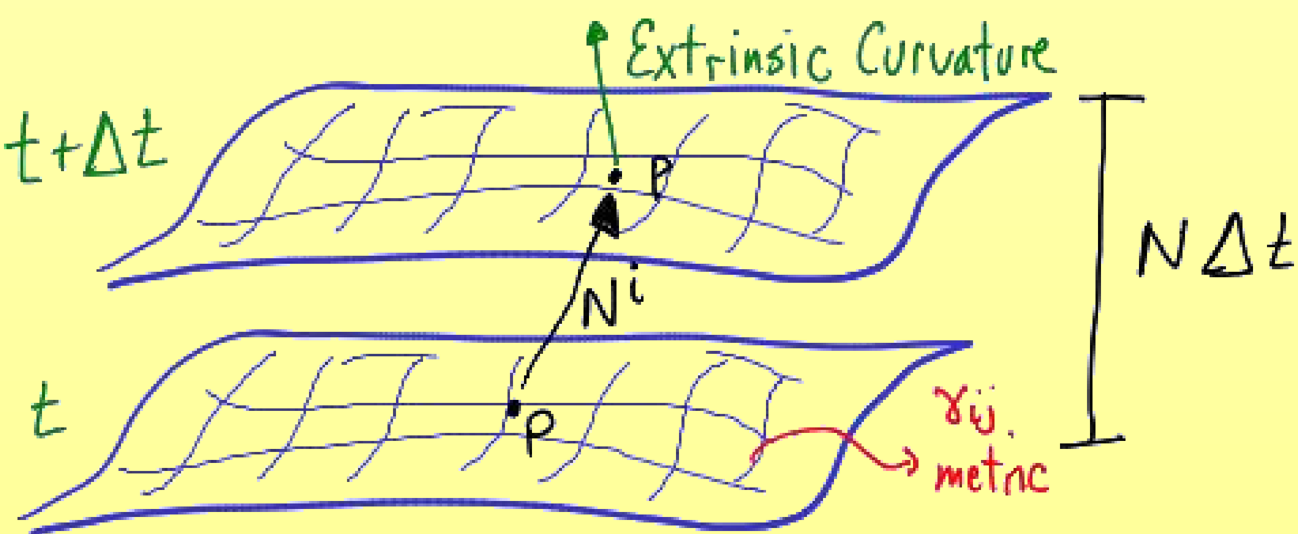
$$g_{\mu\nu} = \begin{pmatrix} -N^2 + \gamma_{ij}N^iN^j & \gamma_{ik}N^k \\ \gamma_{jk}N^k & \gamma_{ij} \end{pmatrix}$$

N lapse function

N^i shift function

g_{ij} spatial metric

Schematically



Cookware: Alternative approaches

Hamiltonian (ADM or 3+1 formalism)

Extrinsic curvature $$K_{ij} = \frac{1}{2N}\left[\partial_t(\gamma_{ij}) - \nabla_i N_j - \nabla_j N_i\right]$$

Then one gets the Hamiltonian

Canonical momenta

$$S = \frac{1}{16\pi G_N}\int \sqrt{-g}d^4x\left(\partial_t\gamma_{ij}\Pi^{ij} - N\ H - N_i\ H^i\right)$$

Hamiltonian $$H = -\sqrt{\gamma}\,{}^{(3)}R + \frac{1}{\sqrt{\gamma}}(K_{ij}K^{ij} - (K^i_i)^2)$$

Momentum $$H^i = 2\nabla_j\Pi^{ij}$$

Cookware: Alternative approaches

Hamiltonian (ADM or 3+1 formalism)

Observations

- Only γ_{ij} is dynamical
- $N,\ N^i$ are Lagrange multipliers which lead to (in vacuum)

$$H \simeq 0, \qquad\qquad H^i \simeq 0$$

Dynamical D.O.F.

6 – (1+3) = 2 ➡ two polarization modes of the graviton

Recipes: linearised theory

Consider perturbation around flat-space

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu} \qquad \epsilon << 1$$

Use the traceless combination

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$$

Using coordinate freedom, choose

$$\nabla_{\mu}\bar{h}^{\mu\nu} = 0$$

Recipes: linearised theory

Consider perturbation around flat-space

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu} \qquad \epsilon << 1$$

GRAVITON

Use the traceless combinatio

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$$

Using coordinate freedom, choose

$$\nabla_\mu \bar{h}^{\mu\nu} = 0$$

Recipes: linearised theory

Then Einstein eqs. reduce to

$$\Box \bar{h}_{\mu\nu} = 16\pi G_N T_{\mu\nu}$$

Or in vacuum

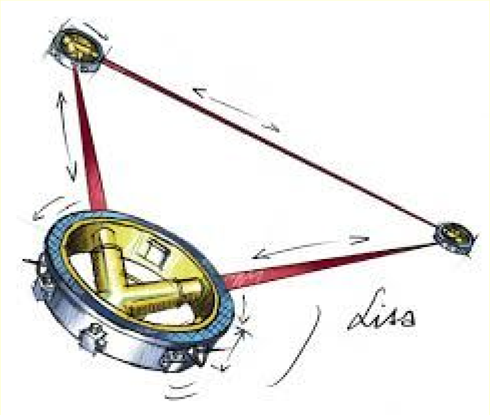
$$\Box \bar{h}_{\mu\nu} = 0$$

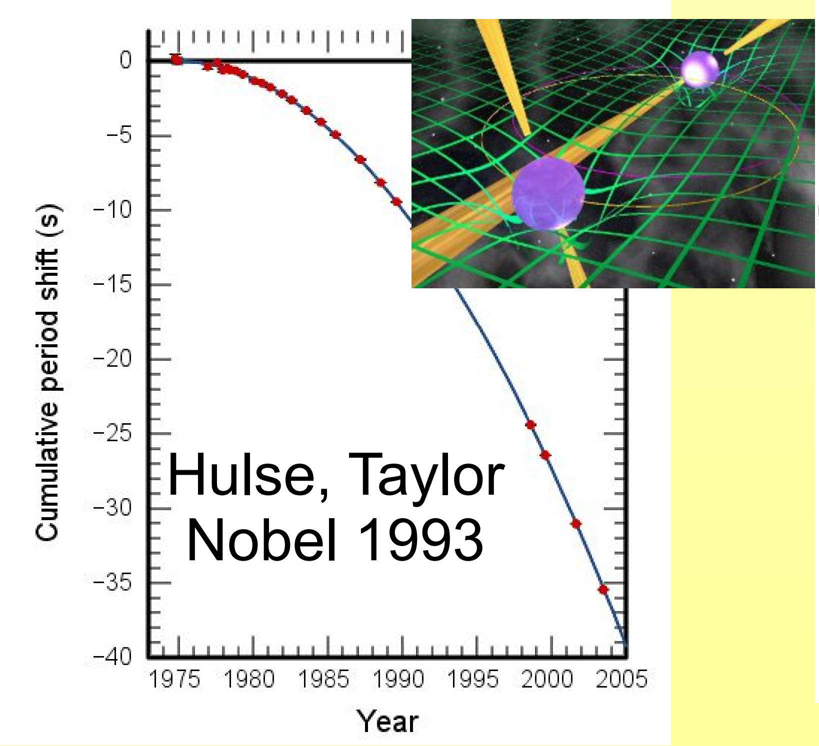
$$\Box = -\partial_t^2 + \nabla^2$$

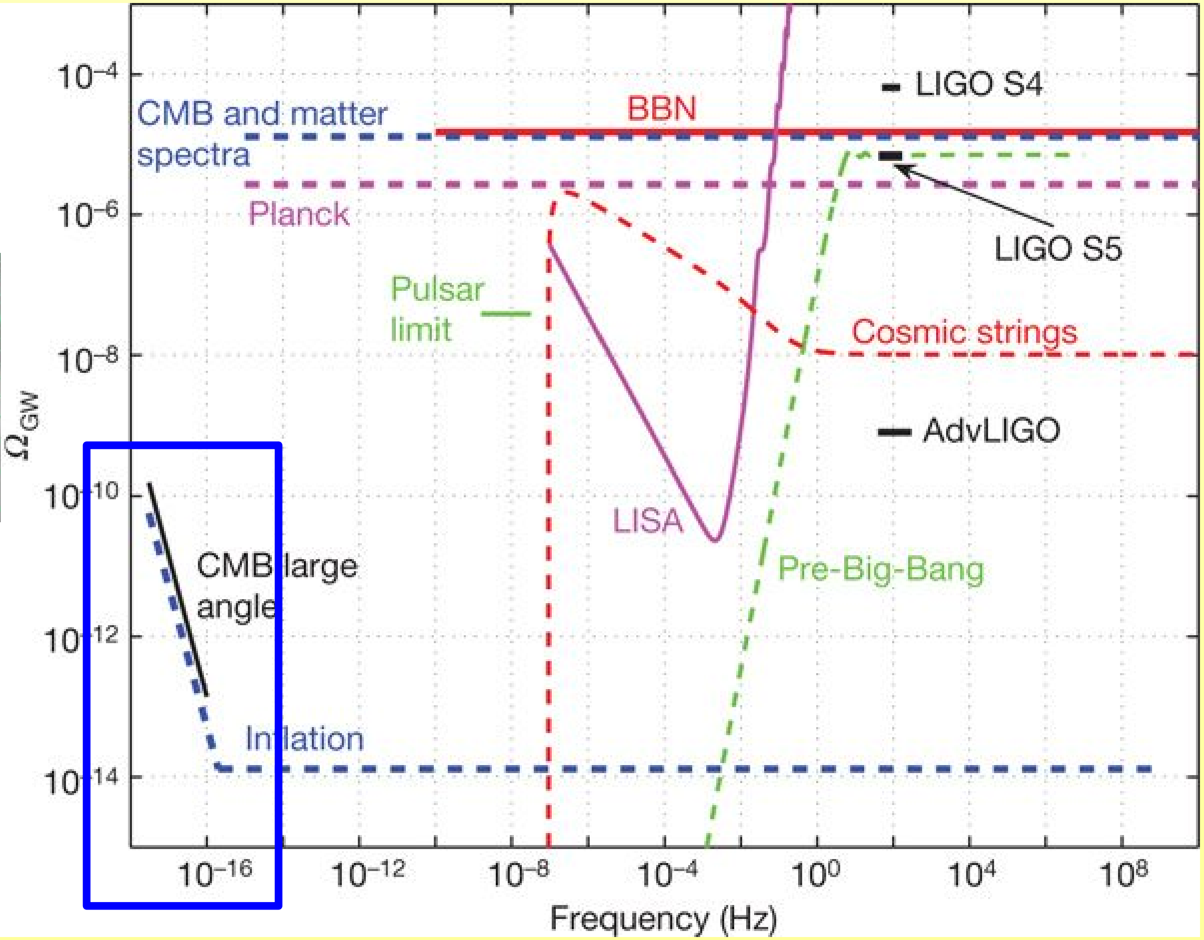
(A wave propagating over Minkowski)

Recipes: Gravity waves

Interferometer's resolution







Recipes: perturbative expansion

Consider the next order in perturbations

$$S = \int \sqrt{-g} d^4x \left(\frac{1}{16\pi G_N} (R - 2\Lambda) + \mathcal{L}_m \right)$$

2nd order in Lagrangian → $\Box \bar{h}_{\mu\nu} = 0$

3rd order in Lagrangian → $\Box \bar{h}_{\mu\nu} = \epsilon (C_1 h^{\mu\nu} h_{\mu\nu} + C_2 h^2)$

Graviton interactions!!!!

Recipes: beyond classical theory

How do we quantise this graviton and its interactions?

$$\Box \bar{h}_{\mu\nu} = \epsilon(C_1 h^{\mu\nu} h_{\mu\nu} + C_2 h^2)$$

QFT

Quantum Field Theory

in a

QFT: review of quantum mechanics

Harmonic Oscillator

$$H \sim p^2 + x^2$$

Define creation/annihilation operators

$$x \sim \sqrt{\hbar}(a^\dagger + a), \qquad p \sim \sqrt{\hbar} i(a^\dagger - a)$$

Hamiltonian becomes

$$H \sim h^2 \left(a^\dagger a + \frac{1}{2} \right)$$

QFT: review of quantum mechanics

Promote ***x*** and ***p*** to operators and impose the Canonical Commutation Relationships

$$[x, p] = i\hbar \quad \Longleftrightarrow \quad [a, a^\dagger] = 1$$

And find that

$$a|n\rangle \sim |n-1\rangle \qquad H|n\rangle = E_n|n\rangle$$

$$a^\dagger|n\rangle \sim |n+1\rangle \qquad E_n = \hbar(n+1/2)$$

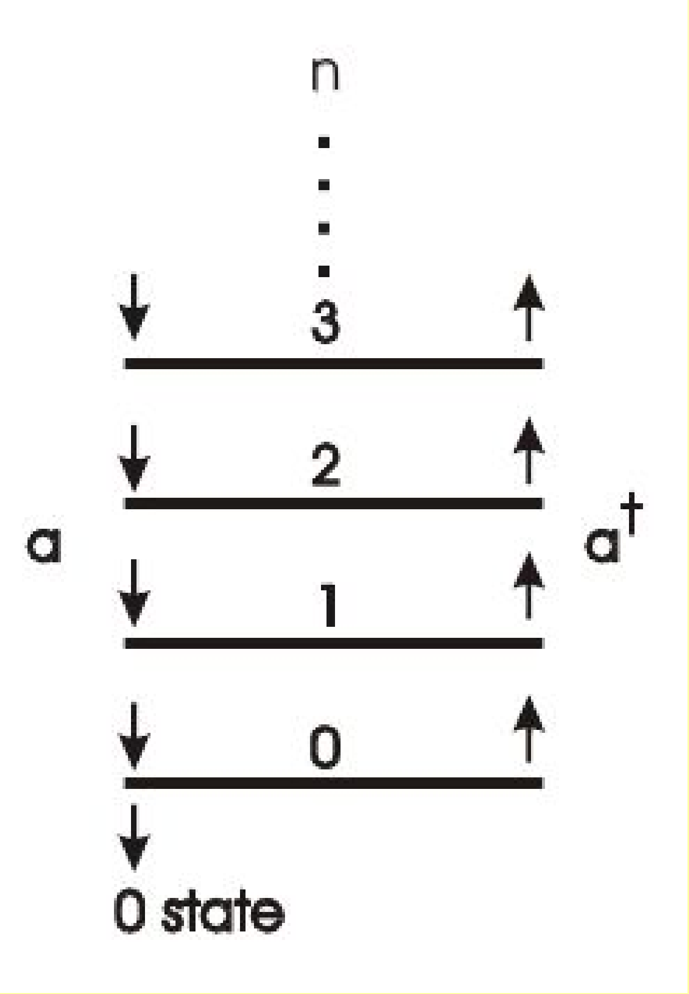
To avoid negative probabilities define vacuum to be

$$a|0\rangle = 0$$

QFT: review of quantum mechanics

Spectrum of states

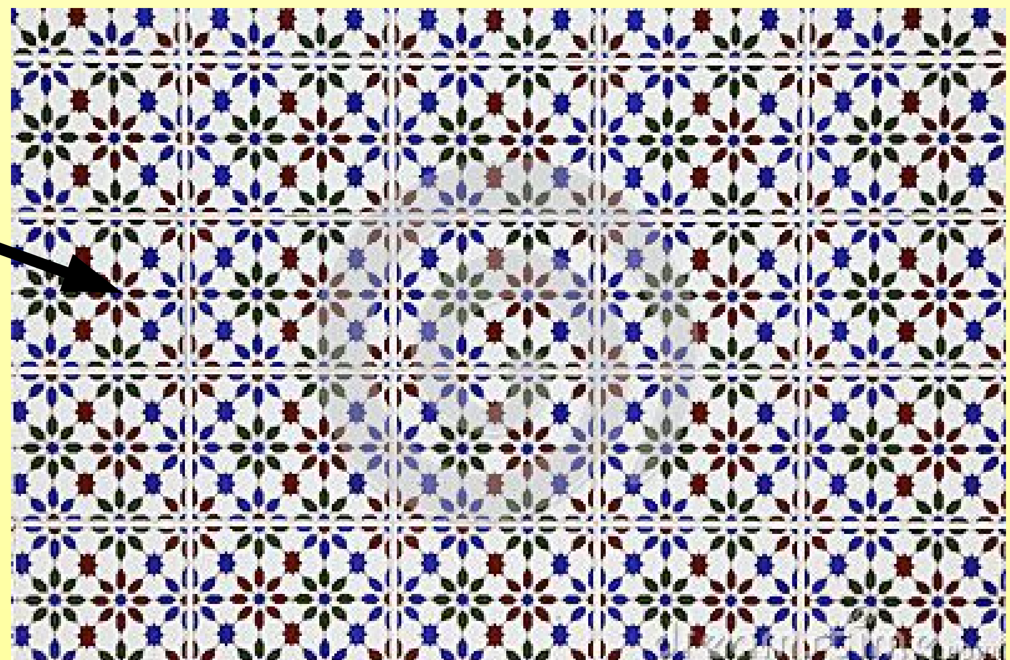
$$E_n = \hbar(n + 1/2)$$



QFT: moving towards fields

Imagine a system with many oscillators, one at each point of space.

$\phi(x, y, z, t)$



Quantum Field Theory!!!!

QFT: moving towards fields

A quantum free (scalar) field...

$$x, p$$

$$[x, p] = i\hbar$$

$$x \sim \sqrt{\hbar}(a^\dagger + a)$$

$$[a, a^\dagger] = 1$$

$$\phi(x, y, z, t), \Pi(x, y, z, t)$$

$$[\phi(x^\mu), \Pi(y^\mu)] = i\hbar\delta(x^\mu - y^\mu)$$

$$\phi(x^\mu) \sim \int dp(a_p^\dagger e^{-ip\cdot x} + a_p e^{ip\cdot x})$$

$$[a_p, a_q^\dagger] = (2\pi)^3\delta(p^\mu - q^\mu)$$

QFT: particles and vacuum

QFT vacuum

$$a_p|0\rangle = 0 \qquad \text{for all } a_p$$

“Particle” of momenta ***p***

$$|p\rangle = a_p^\dagger|0\rangle$$

2 “Particles”

$$|p,q\rangle = a_p^\dagger a_q^\dagger|0\rangle$$

QFT: vacuum energy

Hamiltonian

$$H \sim \int dp \; E_p (a_p a_p^\dagger + a_p^\dagger a_p)$$

$$\sim \int dp \; E_p (a_p^\dagger a_p + (2\pi)^3 \delta(0)/2)$$

Last term diverges ➡ vacuum energy is infinite!

***Solution:* forget about it. Count energy differences**

(normal ordering)

QFT: Propagators

Probability of finding field from point ***x*** to ***y*** $(x^0 > y^0)$

$$\langle 0|\phi(y)\phi(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3 2E_p} e^{ip\cdot(x-y)} \equiv \Delta_F(x-y)$$

Representado por

Can show that

$$\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{-i}{p^2+m^2} e^{ip\cdot(x-y)}$$

QFT: back to the graviton

How do we quantise this massless graviton with interactions?

$$\Box \bar{h}_{\mu\nu} = \epsilon(C_1 h^{\mu\nu} h_{\mu\nu} + C_2 h^2)$$

Simpler case

$$\Box \phi = \epsilon C_1 \phi^2$$

Scalar field over flat (Minkowski's) space-time

QFT: scalar field

Can add a mass (Klein-Gordon equation)

$$(\Box - m^2)\phi = \lambda\phi^2$$

Then Lagrangian is

$$\mathcal{S} = \frac{1}{2}\int d^4x\left[-(\nabla^\mu\phi)(\nabla_\mu\phi) - m^2\phi^2 - \lambda\phi^3\right]$$

$$-V(\phi)$$

Kinetic Term

Potential (interactions)

QFT: scalar field

$$(\Box - m^2)\phi = \lambda\phi^2$$

Classical solution is too hard, but can solve **"free" theory** (LHS only) if $\lambda/E << 1$

In Fourier space

$$\phi = e^{ip\cdot x}\phi_p \quad \text{EOM} \longrightarrow \quad \begin{array}{c} p^\mu p_\mu = m^2 \\ E_p \equiv p^0 = \pm\sqrt{\vec{p}^2 + m^2} \end{array}$$

General soln. is the harmonic oscillator ensemble!

$$\phi(x^\mu) = \int \frac{d^3p}{2\pi\sqrt{E_p}}\left[a_p e^{ip\cdot x} + a_p^\dagger e^{-ip\cdot x}\right]$$

QFT: perturbation theory

Perturbative approach

$$\mathcal{S} = \frac{1}{2}\int d^4x\left[-(\nabla^\mu\phi)(\nabla_\mu\phi) - m^2\phi^2 - \lambda\phi^3\right]$$

EOM $$(\Box - m^2)\phi = 0$$

Now quantise

Propagator $$\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{-i}{p^2+m^2}e^{ip\cdot(x-y)}$$

Inverse of Klein-Gordon

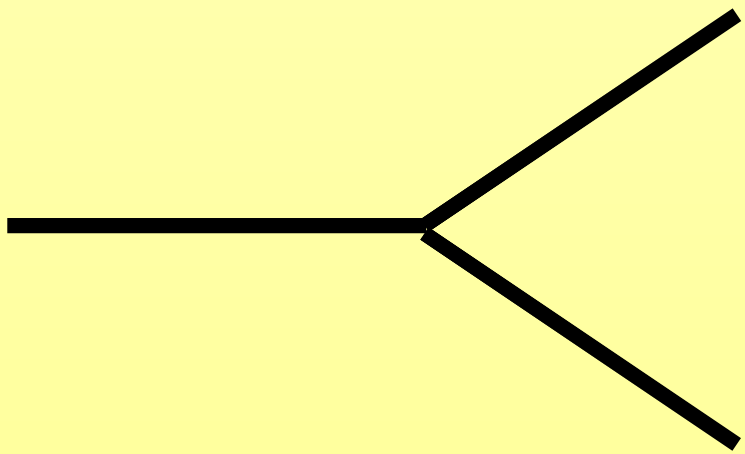
QFT: interactions

Perturbative approach

$$\mathcal{S} = \frac{1}{2}\int d^4x\left[-(\nabla^\mu\phi)(\nabla_\mu\phi) - m^2\phi^2 - \lambda\phi^3\right]$$

Treat interaction as small

Triple vertex

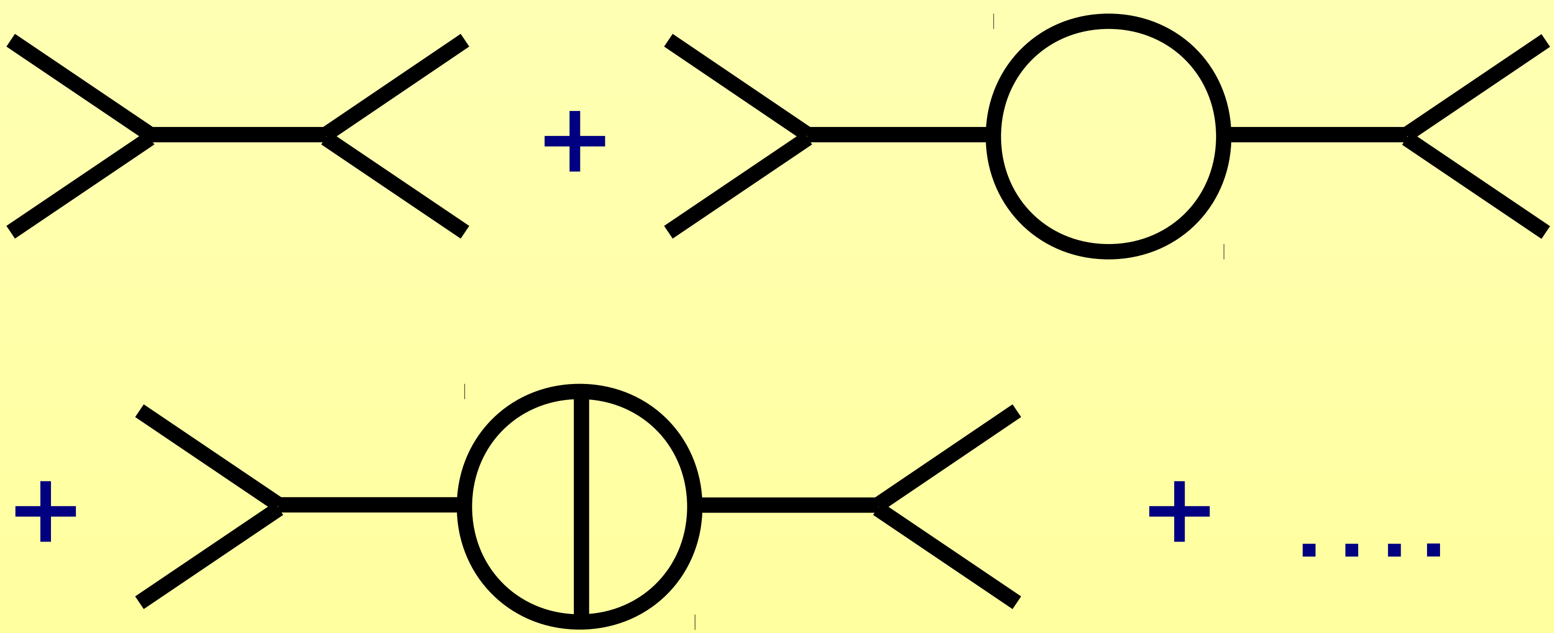


QFT: Feynman diagrams

All possible diagrams contribute to amplitudes

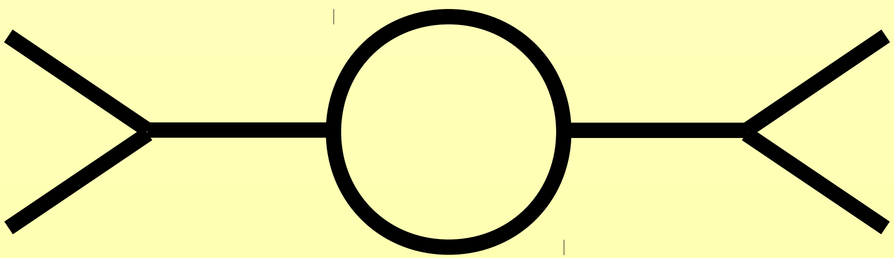
eg. Two particle scattering

$\mathcal{A} \sim$



QFT: Loop divergences

Diagrams like this



may show divergences, like

$$\sim \int_0^\infty \frac{dx}{x+M}$$

Use Renormalisation theory to cure them

QFT: Renormalization

In simple words... consider a convergent series

$$B = \sum_i A_i x$$

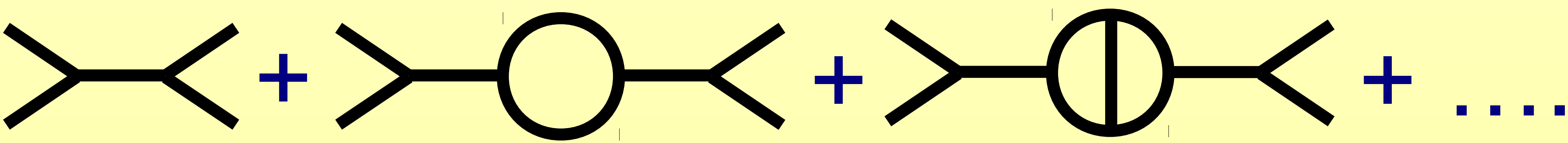
where each A_i is a divergent quantity.

Rewrite the series in new variables ***y*** so that each coefficient C_i is now finite!

$$B = \sum_i C_i y$$

QFT: Renormalization

When doing so for the divergent series



one has to add extra terms into the Lagrangian.

These are called Counter-terms, and depending on their properties one can say if the theory is renormalisable (ie ***if one can remove divergences***)

The UV problem in GR

UV: Non-renormalisable

Where is the problem?

Consider $$S = \int dt d^3x (\partial_\mu \phi \partial^\mu \phi - g \phi^n)$$

Re-scaling $$t \to bt$$

Lorentz invariance

$$x \to bx$$

UV: Non-renormalisable

$$S = \int dt d^3x (\partial_\mu \phi \partial^\mu \phi - g\phi^n)$$

$$\frac{dS}{db} = 0 \quad \Rightarrow \quad \phi \to b^{-1}\phi$$

Potential

$$V = g\int dt d^3 x \phi^n \quad \to \quad b^{4-n} V$$

Diverges in the UV if n>4

UV: Non-renormalisable

Dimensional analysis

$$g \int dt d^3 x \phi^n$$

$$[g] = -(-1 - 3 + n) = 4 - n < 0$$

$$S = \int dt d^3 x (\partial_\mu \phi \partial^\mu \phi - g \phi^n)$$

Propagator $\propto \dfrac{1}{k^2}$

UV: Non-renormalisable

In General Relativity the perturbative theory is

$$\mathcal{L} \sim \sqrt{-g}R \sim \frac{1}{G_N}\left(h^{\mu\nu}\tilde{\Box}h_{\mu\nu} + \mathcal{O}(h^3)\right)$$

$(h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu})$

Differential operator of 2nd order

Can rescale the graviton $h_{\mu\nu} \rightarrow \sqrt{G_N}h_{\mu\nu}$

$$\mathcal{L} \rightarrow \left(h^{\mu\nu}\tilde{\Box}h_{\mu\nu} + \sqrt{G_N}\mathcal{O}(h^3)\right)$$

UV: Non-renormalisable

After rescaling:

$$\mathcal{L} \to \left(h^{\mu\nu} \tilde{\Box} h_{\mu\nu} + \sqrt{G_N} \mathcal{O}(h^3) \right)$$

Free theory

Interactions

Propagator (schematically) $\sim \frac{1}{k^2}$

$$[\sqrt{G_N}] = -1$$

Dimensionful coupling

UV: Non-renormalisable

- We need an infinite number of counter-terms
- Due to Lorentz symmetry the have to be powers of curvature tensors

$$S = \int d^4x (R + \#R^2 + \#R_{\mu\nu}R^{\mu\nu} + \#R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + \ldots$$

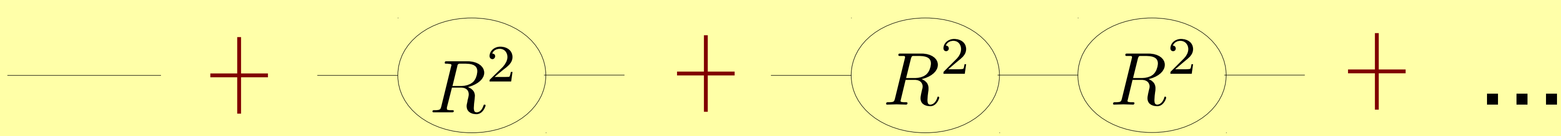
$$\sim \int d^4x h^{\mu\nu}(\tilde{\Box} - G_N \tilde{\Box}^2) h_{\mu\nu} + \ldots$$

UV: Non-renormalisable

So what's wrong with $\int d^4x h^{\mu\nu}(\tilde{\Box} - G_N \tilde{\Box}^2) h_{\mu\nu}$?

Propagator (schematically)

$$\sim \frac{1}{k^2} + \frac{1}{k^2} G_N k^4 \frac{1}{k^2} + \frac{1}{k^2} G_N k^4 \frac{1}{k^2} G_N k^4 \frac{1}{k^2} + \ldots$$



$$\sim \frac{1}{k^2 - G_N k^4}$$

UV: Non-renormalisable

But

Propagator $\sim \dfrac{1}{k^2 - G_N k^4}$

$$\sim \frac{1}{k^2} - \frac{1}{k^2 - 1/G_N}$$

Massive ghost!!

Massless graviton

UV: Non-renormalisable

But

Propagator $\sim \dfrac{1}{k^2 - G_N k^4}$

$$\sim \frac{1}{k^2} - \frac{1}{k^2 - 1/G_N}$$

Massive ghost!!

Massless graviton

Usual problem with higher derivatives (Ostrogradski)

UV: Non-renormalisable

Possible ways out

1) SUSY → SUGRA (helps but not a solution)
2) ~~Lorentz~~
   - Horava-Liftshitz gravity
3) Non-perturbative methods
   - Asymptotic safety
4) Path integral
   - Wave function of the Universe
   - Spin-foams

UV: Non-renormalisable

Possible ways out

4) Canonical quantization (background independent)
   - Wheeler-deWitt
   - Loop quantum gravity
5) New physics/mathematics
   - Algebraic structures (Kac-Moody)
   - String theory
   - Etc.
6) No quantum mechanics at that scale